Intermediation Cost, Credit Expansion and Inequality *

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Abstract

Over the past decades, there has been dramatic credit boom in the United States, coupled with decreasing asset returns and rising inequality. This paper analyzes whether the decreasing intermediation cost of borrowing is an explanation for these developments. It first provides empirical evidence that the intermediation costs have decreased between 1980 and 2007. We then develop a dynamic general equilibrium model of heterogeneous investors and idiosyncratic investment risk. We find that the macroeconomic effects of the fall in intermediation costs are amplified by two feedback loops: one is between the capital market and the credit market, and another is between the capital-credit market and the wealth distribution. We show that, due to lower intermediation costs, the credit market experiences a “simultaneous” expansion of credit demand and credit supply. As a result, the real risk-free interest rate barely changes. The capital market also expands, albeit less significantly than the credit market. The feedback loop between the credit and capital market exaggerates the capital-income risk and increases the average returns on investment among leveraged investors, driving up the overall wealth and income inequality. However, the income inequality among non-leveraged households is reduced, due to the subdued capital-income risk within this group of households. In terms of welfare, we find that the welfare decreases for the households in the bottom-90% wealth group, while it improves for households in other groups.

JEL-Codes: E21, E22, G11, G51

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1 Introduction

Over the past decades, there has been a dramatic credit boom in the United States. Figure 1.1 shows that there is a clear upward trend in the non-financial corporate debt-to-GDP ratio in the United States. From the 1950s until the late 2010s, the corporate credit-to-GDP ratios expanded from around 30% to 75%. Graham et al. (2015) argue that the firm’s characteristics cannot explain the credit expansion, and instead suggest three explanations for the surge in corporate borrowing: changes in government borrowing, macroeconomic uncertainty, and the development of the financial sector. In this paper, we focus on the latter, and analyze how the development of the financial sector affects macroeconomic and distributional outcomes.

Specifically, this paper verifies that the financial development has reduced the intermediation cost of borrowing since the 1980s, and explores whether the fall in the intermediation cost helps explain the credit boom, as well as the following two stylized facts:

A secular rise in income and wealth inequality. The first row of Figure 1.2 plots different measures of the wealth and income distribution of the United States. Since the 1980s, the discrepancies of income and wealth among households have increased. Especially, the top-end inequality has been rising.

Figure 1.1. Non-financial corporate debt-to-GDP of the United States.

![Graph showing non-financial corporate debt-to-GDP ratio from 1952 to 2019.](image)

Source: Bank for International Settlements.
Figure 1.2. Inequality and asset returns in the United States.

Source: The Gini coefficients, as well as the top income and wealth shares are from the World Income Database (WID). The returns of the S&P 500 are from Damodaran (2015). The cyclically adjusted price-earning ratios (CAPE) of the S&P 500 come from Jivraj and Shiller (2017), and the rent-price ratios are from Davis et al. (2008).

A secular fall in asset returns. Over the last four decades, investment has become less rewarding. The second row of the Figure 1.2 illustrates such trend. Stocks and housing are two major assets for investors. As the figure shows, the former market is characterized by decreasing yields, lower risk premium, and increasing prices relative to earnings. The rent-price ratio of the housing market also falls in the long run.

In a first step of this paper, we provide empirical evidence that the development of the financial sector has reduced the intermediation costs of borrowing. The intermediation cost is the part of the
unit cost of debt-finance that is unrelated to the risk premium. These costs are charged by financial intermediaries for their services. Our empirical estimates suggest that, over the period 1980-2007, the average cost of borrowing in the corporate sector of the United States has dropped by roughly 13%-17% due to the reduction in the intermediation cost.

In a next step, we construct a stochastic dynamic general equilibrium model to explore the macroeconomic and distributional consequences of the decrease in the intermediation costs. In our model environment, there are two types of households: leveraged investors (H-types) and non-leveraged investors (L-types). The agents can invest in a risky asset and a safe asset. In our model, the decrease in the intermediation cost triggers a feedback-loop between the capital market and the credit market. A decrease in the intermediation cost makes it cheaper for leveraged H-type investors to borrow, such that they are able to invest more. Higher investment by the H-type investors raises aggregate capital and reduces the return on capital. The decreasing returns on capital induce the non-leveraged L-type investors to reduce their investment and increase their savings. The additional savings by the L-type finance the additional borrowings by the H-type. Hence, the credit market experiences a “simultaneous” expansion of the supply- and demand-side of credit. The additional savings by the L-type agents raise credit supply and encourage the H-type investors to borrow more, which further raises aggregate capital. The falling return on capital induces L-type agents to save more. This feedback loop between the credit market and the capital market amplifies the initial impact of the decreasing intermediation costs. We show that the expansion in the capital market is less pronounced than the expansion in the credit market.

In addition to the feedback loop between the capital and credit market, there exists a second feedback loop between the capital-credit market and the wealth distribution. We show that the decrease in the intermediation cost results in a higher leverage of the H-type, which exaggerates the capital-income risk among the H-type investors and increases their average returns on investment. This drives up the wealth heterogeneity among households. Especially, the tail of the wealth distribution becomes thicker, leading to more rich L-type households who would provide more credit to the H-type investors. This again increases the leverage of the H-type. The feedback between the capita-credit market and the wealth distribution thus further amplifies the initial effect of the decreasing intermediation cost. We find that the decrease in the intermediation costs strongly increases the welfare of the wealthy households. The bottom 90% households, however, face welfare losses.

To highlight the working of the feedback loop between the capital and credit market, we first

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1 For debt-finance through banks, unit cost is equivalent to the net interest spread, i.e., the difference in borrowing and lending rates of the bank. See Section 2 for a detailed discussion on the concept of the intermediation cost.
construct a simple two-period model. We then develop a fully dynamic model based on Angeletos (2007), which we apply to the United States in a quantitative exercise. The model features incomplete markets with heterogeneous agents and idiosyncratic investment risk. Types are Markovian. Following Cao and Luo (2017), in the dynamic model, we assume that the H-type has higher expected returns on investment, which motivates the H-type to leverage. Saving and borrowing are conducted through a perfectly-competitive and risk-neutral financial intermediary. Saving is risk-free, paying out a risk-free rate. The borrower has limited liability. The borrowing interest rate consists of two components, the intermediation cost, which is the spread between the risk-free rate and the lending interest rate to the risk-free borrowers, as well as the default-risk premium, which is determined by the borrower’s type and her leverage. The borrowing interest rate, as well as the borrowing limit is endogenously determined. We conduct counterfactual experiments by varying the intermediation cost. The model has rich implications that fulfill the stylized facts listed above. In the calibrated model, when the intermediation cost of borrowing decreases from 3.0% to 2.0%, credit-output ratio expands from 50.02% to 66.30%. This expansion corresponds roughly to the movement of the US corporate debt-to-GDP ratio from 1980 to 2007 (see Figure 1.1). Because of such a decrease in the intermediation cost, the return on capital falls, and the share of total wealth held by the top 10% (1%) wealthiest households increases from 63.01% (26.63%) to 68.70% (31.51%). In comparison to the data, we find that the change in the intermediation cost explains more than half of the rise in the top-end wealth inequality during 1980 and 2007 (see Figure 1.2). Our quantitative exercise also suggests that the reduction in the intermediation cost does not benefit the bottom-90% households, due to the distribution effect. Therefore, re-distributional policies may be implemented to balance the welfare gains/losses of different wealth groups.

The rest of the paper proceeds as follows. In Section 2, we present empirical evidence of decreasing intermediation cost of borrowing. In Section 3, we develop a simple two-period model to highlight the main theoretical feedback loops between the credit and capital market and the wealth distribution. Section 4 lays out the full model. Sections 5 and 6 discuss the calibration strategy and the quantitative findings. Section 7 concludes.

Related Literature. This paper relates to three strands of literature. First, it adds to the recent discussion of the consequences of credit expansion, such as Iacoviello (2008), Kumhof et al. (2015), etc. While this literature focuses on the expansion of the consumer credit, our emphasis is on the business-credit boom. Graham et al. (2015) documents the upward trend of the corporate debt-to-GDP ratio over the past century, and offers tentative explanations. But they do not provide any discussion of the macroeconomic effects of such an expansion, which is the focus of our paper. There is also growing literature on the macroeconomic consequences of financial deepening, such as
Kiyotaki and Moore (2005), Townsend and Ueda (2006) and Baiardi and Morana (2016). However, the literature on financial deepening is not entirely in line with our topic, for the following two reasons. First, the literature is usually about the emerging economies. For example, Townsend and Ueda (2006) calibrates their model to Thailand, and Dabla-Norris et al. (2014) calibrates to a group of emerging economies. Second, the literature usually emphasizes the link of financial deepening to the economic growth, and to the concept of the “financial Kuznets curve”. Our focus, however, is on the interaction between the credit and capital market.

Second, on the modelling side, this paper builds on the extensive literature on heterogeneous-agent models with incomplete markets, e.g., Aiyagari (1994), Covas (2006), Angeletos (2007), etc. In particular, we build on Angeletos (2007) who presents a heterogenous agent model with incomplete markets and idiosyncratic investment risk. We choose this model as our starting point, because Benhabib et al. (2011, 2015) show that the capital-income risk drives the properties of the right-tail of the wealth distribution. Using Angeletos’ model, but in continuous time, Cao and Luo (2017) discuss the effects of investor’s type persistence on the wealth distribution. Finally, this paper is related to the asset pricing model with incomplete markets by Guvenen (2009).

Finally, this paper is related to e.g., Krusell and Smith (1997), Heaton and Lucas (2002) and Favilukis (2013), who study the properties of portfolio choice and the equity-premium puzzle. One of the main topics on the portfolio choice is the equity-premium puzzle. In our model, we follow this literature and adopt the Epstein-Zin specification of preferences to generate a reasonable return-premium on risky investment. This is based on Epstein and Zin (1989) and Guvenen (2009).
2 Empirical Evidence of Decreasing Intermediation Cost

Intermediation cost is the part of the unit cost of debt-finance that is unrelated to the risk premium.\textsuperscript{2} Figure 2.1 decomposes the cost of borrowing into three components. On the bottom is the risk-free return, which is what the financial intermediaries must pay to savers. Financial intermediation incurs costs to supply credit to borrowers. The cost of credit supply contains the risk-free return and the intermediation cost. Typical examples of the latter include the cost of banks’ office and personnel, screening and monitoring cost on borrowers, as well as the underwriting fees for bond issuance. On top of the credit-supply cost, borrowers also compensate the lenders for the default risk. The cost of borrowing is thus the sum of the risk-free return, the intermediation cost and the risk premium.\textsuperscript{3}

Philippon (2015) estimates the time series of the overall unit cost of external finance for the past 130 years, and finds that the unit cost has not been decreasing in the United States. Eisfeldt and Muir (2016) estimate the average cost of external finance of the US firms from 1980 to 2014 using a structural model. However, both estimates cannot differentiate between the debt- and equity-finance. Moreover, the unit cost contains the risk premium, and the cost of external finance includes the risk-free return and the risk premium. In this sense, they are not perfect measures for the financial efficiency. In contrast to Philippon (2015) and Eisfeldt and Muir (2016), we focus on the intermediation cost of debt finance specifically.

\textbf{Figure 2.1.} Illustration to the intermediation cost and other related costs.

\begin{tikzpicture}
  \tikzstyle{box} = [shape=rectangle, rounded corners, minimum width=3cm, minimum height=1cm, text centered, draw=black]
  \tikzstyle{arrow} = [shape=rectangle, rounded corners, minimum width=1cm, minimum height=0.5cm, text centered, draw=black]
  \tikzstyle{bubble} = [shape=circle, minimum width=0.5cm, minimum height=0.5cm, text centered, draw=black]
  \node (risk) [bubble] {Risk Premium};
  \node (inter) [bubble, below of=risk] {Intermediation Cost};
  \node (risk_free) [bubble, below of=inter] {Risk-free Return (Return to Saver)};
  \node (unit) [arrow, right of=risk_free] {Unit Cost};
  \node (cost) [arrow, below of=unit] {Cost of Credit Supply};
  \draw [arrow] (risk) -- (inter);
  \draw [arrow] (inter) -- (risk_free);
  \draw [arrow] (risk_free) -- (unit);
  \draw [arrow] (unit) -- (cost);
  \draw [arrow] (cost) -- (risk);
\end{tikzpicture}

\textsuperscript{2} The unit cost in Philippon (2015) measures the cost of overall external finance, namely, debt and equity combined.

\textsuperscript{3} The risk premium contains only the price of the default risk. The interest rate risk, however, is not recognized in the risk premium. In our benchmark regression, we use the returns to one-year government bond as a measure of the risk-free rate. The interest rate risk is already priced in such returns.
There are a number of reasons that lead us to believe that the intermediation cost in the credit market has declined over time, such as the development of the IT technology, the increasing credit-information sharing, financial deregulation, credit-market concentration, financial innovation, and the bankruptcy reforms. In this section, we ask the following two questions: (1) does the cost of credit supply in the US corporate sector decrease over time? And if it is true, (2) is the reduction in the intermediation cost behind such decrease. In our empirical analysis, we use a panel data approach and find empirical supports for both questions.

The method that we implement is straightforward. To answer the first question, we apply a two-way error component model with fixed individual and time effects, and regress the cost of borrowing on firm’s characters that potentially affect its borrowing amount and the default-risk premium. Since the risk premium is controlled by firm’s individual characters, negative trend of the estimated time effects can be interpreted as the decrease in the overall credit-supply cost. We find that from the year 1980 to 2007, the average cost of borrowing has dropped by roughly 26%-33% due to the reduction in the cost of credit supply.\footnote{The exact magnitude of the decrease depends on the dataset on which we perform the analysis.} For the second question, we regress the time effects on the real risk-free interest rate. The residuals of the regression measure the movement in the intermediation cost. We find that roughly half of the 26%-33% of the reduction in the borrowing cost is due to the decrease in the intermediation cost.

\subsection{Data and Regression}

\subsubsection{Data}

We use the accounting data from the Compustat North America, which covers all listed companies in the United States and Canada. For our purpose, we download the data for the United States covering the period between 1980 and 2007.\footnote{There are firms that are located in the United States, but with the native currency not being US Dollar. These firms are not included in our sample.} We choose this time window because it is characterized by relatively stable financial-market conditions. In contrast, the 1970s were characterized by the Great Inflation, and the International Financial Crisis and the subsequent Great Recession struck at the end of the year 2007.

The original data is of quarterly frequency, which contain 239,736 firm-quarters. We eliminate 207 observations that have negative entries on at least one of the following balance-sheet items: current assets, total assets, cash and short-term investments, inventories, property, plant, and equipment (net...
of accumulated depreciation), current liabilities, total liabilities, debt in current liabilities, as well as long-term debt. These balance-sheet items are used to define our independent variables, and negative entries on them can be deemed as abnormal. Moreover, we delete 14,009 entries with negative equity (total assets net of total liabilities), and another 19,636 observations due to the abnormality in the dependent variable (see Subsection 2.1.3). Finally, since the intermediation cost is a slow-moving process, and to simplify our empirical procedure, we transform the quarterly data into annual data by averaging (for the balance-sheet items) or summation (for the income-statement or cash-flow-statement items) over quarters. This results in an unbalanced panel that, after data cleaning, consists of 42,383 firm-years. In addition, a balanced panel is constructed by deleting firms that do not have full coverage over the entire time window. The final balanced panel has 11,078 firm-years. Same empirical analysis will be conducted on both balanced and unbalanced panel.

The balanced panel consists of mainly the big and/or long-lasting firms. While in addition to these firms, the unbalanced panel also contains newly-listed firms, as well as firms that have been delisted or terminated. The downside of the unbalanced panel is the existence of a large number of outliers. At the same time, the balanced panel potentially suffers from the firm-age effect. Longer-lasting firms usually enjoy better reputation, which mitigates conflicts of interest between borrowers and lenders (Diamond, 1989). Therefore, the credit availability improves with the firm-age (Petersen and Rajan, 1995), and the borrowing cost decreases with the age of firms (Sakai et al., 2010). In our empirical analysis, the balanced and unbalanced panel serve as robustness check to each other.

2.1.2 Regression

To assess the time effects on the cost of credit supply, we estimate the following two-way error component model with fixed individual and time effects:

\[
Y_{it} = \beta_0 + \mathbf{x}_{it}^T \mathbf{\beta}_1 + \mathbf{x}^{T}_{i(t-1)} \mathbf{\beta}_2 + \mu_i + \gamma_t + \epsilon_{it},
\]

where \(Y\) is the measure of the firm’s borrowing cost. \(\mathbf{x}\) is the vector of controls for the amount of borrowing and for the default premium. In regression model (2.1), we also include the lag-1 controls. This is out of consideration for the fact that the firm’s cost of borrowing relates also to its previous behaviors. \(\mu_i\) and \(\gamma_t\) are fixed individual and time effects, respectively. One of the advantages of using

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6 See Appendix A.2 for the illustration of the existence of outliers in the unbalanced panel. Our methodology is to manipulate the dataset as little as possible, hence we do not kick the “abnormal” firms out of the unbalanced panel. Rather, we compare the regression results of the unbalanced panel with that of the balanced panel, which do not have the issue of outliers. In Subsection 2.2 and Appendix A.3, one will see that the existence of outliers in the unbalanced panel does not affect our empirical conclusions.
the fixed-effect model is that we do not have to control for the cross-sectional firm-characters, such as the firm’s industry, sector, or geography. The term $\mu_i$ absorbs all the cross-sectional effects on the dependent variable. The OLS estimation of the model only accounts for the “within-group” effect, i.e., the variation across time for each individual firm. Finally, $e_{it}$ is the identically and independently distributed (i.i.d.) error term.

As the variation of the borrowing amount and the risk premium are covered by the controls, the time effects $\gamma_t$ thus measure the effects of the change in the overall credit-supply cost on the cost of borrowing. To further disentangle the time effects of the intermediation cost from that of the credit-supply cost, we regress the $\gamma_t$ on the real risk-free interest rate $r_t$

$$\gamma_t = \alpha_0 + \alpha_1 r_t + \varepsilon_t,$$  \hspace{1cm} \hspace{1cm} (2.2)

where $\varepsilon_{it}$ is the identically and independently distributed (i.i.d.) error term. The estimates of $\varepsilon_t$ is orthogonal to $r_t$. Hence, it measure the effect of the change in the intermediation cost on the cost of borrowing.

2.1.3 Variables

We consider a measure of the firm’s borrowing cost as the dependent variable in our regression model (2.1). We choose the Compustat item “Interest and its related expense” as such measure. The item incorporates all the costs related to the debt finance, of which the intermediation cost is a part. To preclude the influence of inflation, we scale the dependent variable by revenue.\textsuperscript{7} Let $Y$ be the dependent variable,

$$Y = \frac{\text{Interest and its related expense}}{\text{Revenue}} \times 100. \hspace{1cm} \hspace{1cm} \text{(2.3)}$$

Reasonable range of $Y$ should be within 0 and 100. $Y$ larger than 100 indicates that the firm’s revenue is no longer able to compensate for the interest expenses; while a negative $Y$ implies that the firm is a net lender. We deem both case as abnormal. Observations with $Y$ not in the range of $(0, 100]$ are thus excluded from both of the panels.

\textsuperscript{7} Scaling the cost of borrowing also facilitates the use of accounting ratios as control variables (see Table A.1). These accounting ratios are widely accepted tools for analyzing firm’s financial status, and they are directly behind the lender’s perception of the firm’s financial soundness. We choose the revenue as the scaling factor for two reasons. First, same as the interest expense, revenue is also a flow variable; and second, the revenue is relatively stable over time, compared to other flow items such as operating income, earnings, etc.
Figure 2.2. The cross-sectional means of the dependent variable (left) and the fixed time effects on the borrowing cost (right).

The left panel of Figure 2.2 plots the means of the dependent variable across firms for each year. The cost of borrowing represents 4%–7% of revenues on average. In the balanced panel, the average cost of borrowing per unit of revenue decreases over time. However, we do not observe a downward trend in the unbalanced panel.

Detailed definitions of the control variables in $x$ are provided in Table A.1 of the Appendix A. To specify the control variables, we seek possible factors that can explain the firm’s revenue, amount of borrowing, as well as its risk premium over time. These factors can be grouped into six categories: liquidity, capital structure, solvency, profitability, market valuation and others (size, tangibility, as well as the percentage change in the long-term and short-term debt-to-equity ratio). The controls are formulated as accounting ratios to preclude the influence of inflation. These accounting ratios are widely applied to assess firm’s financial soundness. Subsection A.1 of the appendix provides a detailed discussion on the control variables. Appendix A.2 summarizes the descriptive statistics of the dependent and independent variables. Finally, in the regression model (2.2), the real interest rate $r_t$ is measured by the one-year treasury constant maturity rate, deflated by the CPI.\(^8\)

\(^8\) There exist other measures of the risk-free rate and the inflation. Among the risk-free rates, one can also use the federal fund rate and US dollar-denominated LIBOR rate. These interest rates are highly correlated with each other, and thus in principle should lead to similar regression results. In Appendix A.3.2, we show that using different measures of the risk-free rate and the inflation rate (CPI, PCE and the GDP deflator) do not change our empirical results.
Notes: Each scatter point is the estimated time effect $\hat{\gamma}_t$ of the model (2.1). The linear time effect is the estimated regression coefficient $\hat{\eta}_1$ in the simple regression model $\hat{\gamma}_t = \eta_0 + \eta_1 \times T_t + \epsilon_t^{\Delta}$, where $T_t = \text{calendar year} - 1982$, and $\epsilon_t^{\Delta}$ is the residual. *** denotes significance at the 1%.

2.2 Results

The regression model (2.1) is estimated using ordinary least squares (OLS). As our focus is on the time effects, we move the discussion on the regression results of the controls to Appendix A.3. The estimates of the time effects $\hat{\gamma}_t$ are plotted in the right panel of Figure 2.2. As we can see, over time, the time effects on the borrowing cost are consistently negative. This is the case for both panels. On average, the borrowing cost has plunged by 1.6% of the revenue between 1980 and 2007. Note that the number of 1.6% is the absolute change of the dependent variable. Because the dependent variable has a mean of 4.86% in the balanced panel and 6.26% in the unbalanced panel (see the left panel of Figure 2.2), the decrease in the borrowing cost is approximately 26%-33% in relative terms, which is a substantial reduction.

In the regression model, we have controlled for the firm characteristics that potentially affect the revenue, amount of borrowing and the risk premium. Therefore, the fixed time effects contain only the unexplained factor of the credit-supply cost (recall Figure 2.1). Our estimation results suggest that the cost of credit supply has been constantly decreasing over time, and the drop in the credit-supply cost
Figure 2.4. Fixed time effects of the intermediation cost.

Notes: Each scatter point is the estimated residual $\hat{\varepsilon}_t$ of the model (2.2). The linear time effect is the estimated regression coefficient $\hat{\delta}_1$ in the simple regression model $\hat{\varepsilon}_t = \delta_0 + \delta_1 \times T_t + e_t^0$, where $T_t = \text{calendar year} - 1982$, and $e_t^0$ is the residual. *** denotes significance at the 1%.

contributes to the 1.6% decrease in the borrowing cost over revenue. In Figure 2.3, we fit a linear trend to the estimated time effects $\hat{\gamma}_t$. The linear trend implies that the borrowing cost over revenue drops roughly by 0.06% (absolute change) each year over the period 1980 and 2007, due to the reduction in the cost of credit supply. In relative terms, the borrowing cost drops by roughly 1.0%-1.2% annually because of the change in the cost of credit supply.

By estimating the model (2.2), time effects of the intermediation cost can be disentangled from that of the credit-supply cost. The estimation results are reported and discussed in Appendix A.3. Here we only focus on the estimates of the residuals. As has been discussed above, the residuals of the regression measure the effects of the decrease in the intermediation cost on the cost of borrowing. In Figure 2.4, we fit a linear trend to the estimated residuals $\hat{\varepsilon}_t$. As we can see, in absolute terms, the change in the intermediation cost contributes to the drop in the cost of borrowing by 0.84% of the revenue, corresponding to an annual decrease of the latter by 0.035% from 1980 to 2007. In other words, the drop in the intermediation cost and the reduction in the real risk-free rate contributes roughly equally to the 1.6% decrease in the borrowing cost over revenue. We thus conclude that the intermediation cost has indeed decreased significantly between 1980 and 2007.
In Appendix A.3.2 we explore the regression results of the model (2.2) if we use the Aaa-graded corporate bond yield as the interest rate. Because the corporate bond yield already contains the intermediation cost, if our setup is correct, the regressor should fully cover the variation in the dependent variable. In this case, the residuals of the regression should not have a significant trend. Our regression results confirm that this is indeed the case.\textsuperscript{10} When using the corporate bond yield as regressor, the linear time effect of the residuals is not significant any more. For both panels, the linear time effect is close to zero. This justifies our results and methodology. For more discussion on the robustness check see Appendix A.3.2.

\textsuperscript{10} See Table A.7 of Appendix A.3.2 for the estimation results.
3 A Two-Period Model

In this section, we set up and analyze a two-period model to assess the macroeconomic implications of the decrease in the intermediation cost. In particular, we evaluate the properties and consequences of the feedback loop between the capital and the credit market. The two-period model is built up for explanatory purposes, since it can be solved analytically. As it turns out, many complicated relationships in the dynamic model are straightforward in the two-period model. The conclusions of the two-period model are the same as what we obtain in the dynamic model of the next section. We hence advise the reader who is impatient to skip this section, and directly start from the Section 4.

3.1 The Model Environment

The economy consists of an infinite number of agents. The total population of the economy is normalized to one. In the economy, there is a share of $p$ agents that are of type H, and the rest $(1-p)$ are of type L. The economy lasts for two periods, $t = 0, 1$, and agents only consume at the end of period $t = 1$.\footnote{Alternatively, one can assume an infinite-horizon version of the model, in which the period $t = 0$ corresponds to the morning of day, and $t = 1$ corresponds to the afternoon of the day. If this assumption is adopted, one has to assume in addition that all endowment and production cannot be stored, so that they must be consumed at the end of the day.}

Each agent obtains endowment $w$ with different timings: the L-type receives the endowment at the beginning of period $t = 0$, and the H-type receives the endowment at the beginning of period $t = 1$. The different timing across types has two consequences: first, it “artificially” creates leveraged (H-type) and non-leveraged investors (L-type). Second, the H-type will be essentially risk-neutral.\footnote{The conclusions of the model do not depend on the specific form of the type-difference, as long as the H-type has higher motivation to invest than the L-type. For example, the two-period model has the same implications as the dynamic model of the next section, although the latter assumes a different form of type-difference. Moreover, even the extreme version of the type-difference assumed in the two-period model has some support from the literature. There are empirical findings that the investors (i.e., asset market participants) have lower risk aversion and higher EIS, see, for example, Guvenen (2006), and our set-up fulfills such empirical regularity.}

For simplicity, agents are assumed to have log-preferences over consumption.

Agent can invest to produce the consumption good. Investment is risky, and takes place only at the beginning of period $t = 0$. Denote by $i_L$ and $i_H$ the investment of the L-type and H-type, respectively. At the end of period $t = 0$, there is a capital formation process, which transforms investment of either type into productive capital,

\[
k_L = \varepsilon i_L \quad \text{and} \quad k_H = \varepsilon i_H,\tag{3.1}
\]
where $k_L$ and $k_H$ denote capital of the L-type and the H-type, respectively. $\varepsilon$ is a Bernoulli random variable, which takes the value 1 with probability $1 - \lambda$. The realization of $\varepsilon$ is idiosyncratic to each agent. In other words, if an agent invests, there is a probability of $\lambda$ that the investment fails and all investment of the agent is erased. By the law of large numbers, at the end of period $t = 0$, the total amount of capital in the entire economy is

$$K = (1 - \lambda) \left[ p_i H + (1 - p) i_L \right].$$  \hspace{1cm} (3.2)

It is assumed that when investing and not investing result in the same utility, the agent always invests.\textsuperscript{13}

At the beginning of period $t = 1$, there is a perfectly competitive representative firm, that pools the aggregate capital produced in the last period and produces the consumption good to be consumed at the end of the period. The production function is

$$Y = K^\alpha,$$  \hspace{1cm} (3.3)

where $\alpha \in (0, 1)$. The gross return to capital is $R = Y/K = K^{\alpha - 1}$. The capital return is distributed according to the capital holdings of the agents. The timing of events of the economy is summarized in Figure 3.1.

There is a credit market in the economy. The credit market is intermediated by a perfectly-competitive representative bank. The saving to the bank is risk-free, with interest rate $r_f$ that is determined by the market-clearing condition. If an agent borrows, there is an intermediation cost proportional to the amount of borrowing. The intermediation cost is captured by the parameter

\textbf{Figure 3.1.} Timing of events in the two-period economy.

---

\textsuperscript{13} Equivalently, we can also assume that investing brings a tiny utility gain of “staying-in-business”.
\( \theta \in (0, 1) \). When the agent borrows \( b \), she can apply \( \frac{1}{1+\theta} b \) to investment, while the rest \( \frac{\theta}{1+\theta} b \) is the intermediation cost which is wasted. In the model, agent borrows for investment,\(^{14}\) and she is prohibited from simultaneously borrowing and saving.\(^{15}\) Borrower has limited liability, namely, the agent’s borrowing is only secured by her investment.\(^{16}\) The bank sets the interest rate \( \pi \) on borrowing according to the zero-profit condition. In particular, the bank makes zero profit after charging the interest rate from the borrowers and paying for the risk-free interest to the savers.

### 3.2 General Equilibrium

Since the H-type does not own resources at the beginning of \( t = 0 \), she has to borrow to invest. Because of the existence of the intermediation cost, to invest \( i_H \), she has to borrow \( b = (1 + \theta)i_H \). The bank is risk-neutral and perfectly competitive. If the bank is willing to lend, it charges the interest rate \( \pi \) according to the zero-profit condition

\[
(1 + r^f)b = (1 - \lambda)(1 + \pi)b,
\]

where \( r^f \) is the risk-free interest rate. Hence, the interest rate is given by

\[
1 + \pi = \frac{1 + r^f}{1 - \lambda}.
\]

Lemma 3.1 states that in equilibrium, the bank is always willing to lend to the H-type.

If the bank is willing to lend and the H-type decides to borrow, the latter’s optimization problem can be formulated as

\[
\max_{i_H} \left\{ (1 - \lambda) \log \left( R^c i_H - (1 + \pi)b + w \right) + \lambda \log w \right\},
\]

\[ s.t. \quad b = (1 + \theta)i_H \quad \text{and} \quad i_H \geq 0, \]

where \( R^c i_H - (1 + \pi)b + w \) is the wealth (consumption) that the H-type obtains if the capital-formation is successful. Remember that the endowment cannot be used as a commitment to borrowing. If the

---

\(^{14}\) Since the agent has only one-time consumption at the end of \( t = 1 \), consumer credit is not possible in the model.

\(^{15}\) Allowing for simultaneous saving and borrowing does not change any conclusion, but it complicates the analysis. In the model, only the L-type agent has the chance to save and borrow simultaneously. However, it can be shown that, in the equilibrium, it is not optimal for her to do so.

\(^{16}\) We can imagine that each agent invests by setting up a limited-liability company. Even in the worst case, the bank can only grab the company’s asset. The agent’s endowment, if not being put into investment, cannot be claimed by the bank.
capital formation fails and her investment is wiped out, the H-type can still consume her endowment \( w \) in the end. The optimization problem (3.6) implies that the H-type is essentially risk-neutral; and she keeps investing until the marginal returns from investment equals the marginal cost,

\[ R^c = (1 + \pi)(1 + \theta). \tag{3.7} \]

In equilibrium, the H-type makes zero profit from investment. But because the agent prefers “staying-in-business”, H-type still invests. The condition (3.7) builds a constant relationship between the capital return and the risk-free rate

\[ (1 - \lambda)R^c = (1 + \theta)(1 + r^f), \tag{3.8} \]

which implies a risk premium of \((1 - \lambda)R^c - (1 + r^f) = \theta(1 + r^f)\). The risk premium equals the opportunity cost of \(\theta\), which is the consequence of the risk-neutrality of the H-type.

The H-type is the net borrower in the credit market. If the credit market does not collapse, the L-type must be a net saver. Namely, the L-type must allocate her endowment between risky investment and risk-free saving. If the credit market exists, the L-type solves the following problem:

\[
\max_{i_L} \left\{ (1 - \lambda) \log \left[ (1 + r^f)s + R^ci_L \right] + \lambda \log \left[ (1 + r^f)s \right] \right\},\tag{3.9}
\]

\[ \text{s.t. } s = w - i_L \quad \text{and} \quad i_L \geq 0, \]

where \(s := w - i_n\) is the amount of saving by the L-type. The first-order condition of the optimization problem (3.9) implies that

\[ i_L = \left[ 1 - \frac{\lambda R^c}{R^c - (1 + r^f)} \right] w. \tag{3.10} \]

Lemma 3.1 In equilibrium, credit market exists and \(0 < i_L < w\).

Proof see appendix.

We can now discuss the behavior of the H- and L-type when facing the decrease in the intermediation cost \(\theta\). Suppose that the economy is initially in equilibrium and condition (3.7) holds. When \(\theta\) decreases, \((1 + \pi)(1 + \theta)\) becomes smaller than \(R^c\). Thus the first-order derivative of the H-type’s objective function (3.6) becomes strictly positive. The H-type increases investment \(i_H\) until condition (3.7) binds again. Proposition 3.2 presents the solution to the general equilibrium of the economy. In general equilibrium, the increase in H-type’s leveraged investment drives up the risk-free rate \(r^f\) and drives down \(R^c\). According to equation (3.10), such changes in \(r^f\) and \(R^c\) reduce \(i_L\). Thus the L-type invests less and saves more when \(\theta\) decreases.
\( f^f, R^c, \) as well as the aggregate capital stock \( K \) are determined in the general equilibrium. In the general equilibrium, credit market, good market, as well as the capital market clears. The general equilibrium of the economy is concluded in the following proposition.

**Proposition 3.2 (General Equilibrium)** *In the general equilibrium,*

\[
i^f_L = \left[ 1 - \frac{\lambda(1 + \theta)}{\theta + \lambda} \right] w; \quad (3.11)
\]
\[
i^f_H = \frac{1 - p}{p} \cdot \frac{\lambda}{\theta + \lambda} w; \quad (3.12)
\]
\[
b = (1 + \theta)i^f_H = \frac{1 - p}{p} \cdot \frac{\lambda(1 + \theta)}{\theta + \lambda} w; \quad (3.13)
\]
\[
s = w - i^f_L = \frac{\lambda(1 + \theta)}{\theta + \lambda} w. \quad (3.14)
\]

Moreover, the aggregate capital capital of the economy is

\[
K = (1 - p)(1 - \lambda) \left[ 1 - \frac{\lambda \theta}{\theta + \lambda} \right] w, \quad (3.15)
\]

*with the gross return to capital \( R^c = K^{\alpha - 1} \). The total intermediated asset \( S \) and risk-free interest rate \( r^f \) are*

\[
S = (1 - p)\frac{\lambda(1 + \theta)}{\theta + \lambda} w; \quad (3.16)
\]
\[
1 + r^f = \frac{1 - \lambda}{1 + \theta} R^c. \quad (3.17)
\]

**Proof** see appendix. ■

When the intermediation cost \( \theta \) decreases, \( K \) increases in equilibrium, and \( R^c \) drops accordingly. This means that when \( \theta \) decreases, the additional investment by the H-type drives down \( R^c \). Decreasing \( R^c \) propels the L-type to reduce her investment in risky capital and increase her saving (equation (3.11)). Therefore, in the credit market, the L-type saves more to finance more borrowing of the H-type. Hence the total intermediated asset expands (equation (3.16)). We call this expansion a “simultaneous” credit expansion, because such an expansion is driven by both the supply (L-type) and demand side (H-type).

The feedback between the capital and the credit market amplifies the initial effect of the decrease in the intermediation cost. The additional saving by the L-type finances further leveraged investment of the H-type. Thus, \( i^f_H \) increases again, while \( R^c \) and \( i^f_L \) drops further. In the end, the L-type supplies even more credit. The feedback loop knocks on until new equilibrium is reached. In the remainder of this section, we will discuss the consequences of the feedback loop on the risk-free rate, inequality and the welfare.
3.3 Risk-free Rate

In equation (3.17), \( R^c \) and \( \theta \) have opposing effect on \( r^f \). In response to a decrease in \( \theta \), the change of \( r^f \) depends on the strength of credit-supply expansion relative to the credit-demand expansion. In the two-period model, it can be easily verified that when \( \theta \) drops, the credit-demand effect dominates the credit-supply effect, and the risk-free rate \( r^f \) increases.

**Proposition 3.3** When \( \theta \) decreases, \( r^f \) increases in equilibrium.

**Proof** see appendix. ■

The increasing risk-free rate comes from the fact that the H-type is risk-neutral and has a higher motivation to invest, compared to the L-type. Therefore, the H-type’s aggregate investment response to the decreasing intermediation cost is stronger than that of the L-type. To see this, let \( \mathcal{I}_H \) and \( \mathcal{I}_L \) denote the aggregate investment of the H-type and L-type, respectively, so that \( \mathcal{I}_H = p\hat{i}_H \) and \( \mathcal{I}_L = (1-p)i_L \). Based on equations (3.11) and (3.12) we have

\[
\frac{\partial \mathcal{I}_H}{\partial \theta} = -\frac{\lambda (1-p)}{(\theta + \lambda)^2} w, \quad \text{and} \quad \frac{\partial \mathcal{I}_L}{\partial \theta} = \frac{\lambda (1-\lambda)(1-p)}{(\theta + \lambda)^2} w.
\]

Therefore, we can see that

\[
\left| \frac{\partial \mathcal{I}_H / \partial \theta}{\partial \mathcal{I}_L / \partial \theta} \right| = \frac{1}{1-\lambda} > 1,
\]

where the last inequality holds because \( 0 < \lambda < 1 \). This implies that when \( \theta \) drops, the increase in the aggregate investment of the H-type is faster than the decrease in that of the L-type. Therefore, the aggregate capital \( K \) increases, which results to a lower capital return \( R^c \). On the other hand, remember that the H-type’s investment is borrowed, and the reduced amount of the L-type’s investment is put into savings. Equation (3.18) thus also implies that when \( \theta \) decreases, the increase in the aggregate credit demand is faster than the increase in the aggregate credit supply. \( r^f \) increases as a result.

3.4 Credit Expansion versus Capital Expansion

When the intermediation cost \( \theta \) decreases, both the capital market and the credit market expand. But the strength of the credit expansion surpasses that of the capital expansion. To see this, let’s first compare aggregate savings and aggregate capital in general equilibrium. By equations (3.15) and (3.16),

\[
\frac{S}{K} = \frac{\lambda}{1-\lambda} \cdot \frac{1+\theta}{\lambda + (1-\lambda)\theta}.
\]
The ratio of $S/K$ increases when $\theta$ decreases, because
\[
\frac{\partial (S/K)}{\partial \theta} = -\frac{\lambda}{1-\lambda} \cdot \frac{1}{\left[ \lambda + (1-\lambda) \theta \right]^2} < 0. \tag{3.19}
\]
Equation (3.19) implies that when $\theta$ drops, $S$ increases more than $K$. The reason behind this finding is easy to understand. When $\theta$ decreases, the credit market experiences a “simultaneous” expansion from both demand and supply side. However, in the capital market, the increase in investment by the H-type (which equals exactly to the increase in the credit demand) is dampened by the decrease in investment by the L-type. Hence the capital expansion is less significant than the credit expansion.

3.5 Inequality

The model has implications on distributions. Since the model has only two periods, wealth, income and consumption can be treated as the same. The H-type, no matter whether the capital formation is successful and what the level of $\theta$ is, always consumes $w$ in the end. Hence only the L-type bears the differences in income. Denote by $c^1_L$ and $c^0_L$ the L-type’s consumption when the capital formation is and is not successful, respectively. Namely, $c^1_L = (1+r^f)s + R^c i_L$ and $c^0_L = (1+r^f)s$. In equilibrium, by using equations (3.11), (3.14) and (3.17), we have
\[
\begin{align*}
    c^1_L &= (1-\lambda)R^c w; \\
    c^0_L &= \frac{\lambda}{\theta + \lambda} c^1_L. \tag{3.21}
\end{align*}
\]
Since $R^c$ is decreasing in $\theta$, $c^1_L$ decreases when the intermediation cost drops. It can be easily verified that the opposite is true for $c^0_L$. Therefore, the lower intermediation cost decreases the L-type’s consumption (income) when the capital formation succeeds, and increases its consumption (income) when the capital formation fails. Figure 3.2 summarizes the income distribution among the L-types. In the two-period model, the change in L-types’ income distribution is equivalent to the change in their capital-income risk.\footnote{To be consistent with the existing literature, we define the “capital income” as the income from both risky investment and risk-free saving.} To see this, we simply calculate the difference between $c^1_L$ and $c^0_L$: $c^1_L - c^0_L = R^c i_L$. In the general equilibrium, both $i_L$ and $R^c$ decreases when $\theta$ decreases. Hence the L-type’s income difference between when capital-formation success and when capital-formation fails becomes smaller. This can be deemed as the decrease in the capital-income risk, which is equivalent to a concentration of income among the L-types.
Figure 3.2. Change of income distribution when the intermediation cost decreases.

<table>
<thead>
<tr>
<th>H-type</th>
<th>L-type fails</th>
<th>L-type success</th>
</tr>
</thead>
<tbody>
<tr>
<td>share $p\lambda$</td>
<td>share $\lambda(1-p)$</td>
<td>share $(1-\lambda)(1-p)$</td>
</tr>
</tbody>
</table>

$c_H = w \quad c_L^0 \quad c_L^0' \quad c_L^1 \quad c_L^1'$

Notes: When $\theta$ drops, the L-type’s consumption changes from $c_L^1$ to $c_L^1'$ when the capital-formation succeed; and from $c_L^0$ to $c_L^0'$ when it fails.

The two-period model cannot differentiate between the income and wealth distribution. Moreover, the model’s implication that the H-type’s income is less than the L-type’s is also counter-intuitive. These issues will be addressed in the dynamic model of the next section. But the main takeaway from the discussion in this subsection is the relationship between the capital-income risk and the (income) inequality. Among other things, the income inequality among L-types drops also in the dynamic model, due to the same reason as what we have seen in the two-period model.

3.6 Welfare

Because the H-type is risk-neutral and always makes zero profit, the change of the overall ex ante welfare depends on the L-type. It can be shown that with $\theta$ decreasing, the ex ante welfare of the L-type is improving.

Proposition 3.4 *Decreasing $\theta$ improves the ex ante welfare of the economy.*

Proof see appendix.

It can be easily verified that in the general equilibrium, when $\theta$ drops, $S$ increases but $\theta S$ decreases. This means that with lower $\theta$, financial sector consumes less resources, although the total quantity of the intermediated asset increases. The improved welfare thus comes from the more cost-efficient financial sector.
4 The Dynamic Model

This section presents the dynamic model that highlights the macroeconomic effects of the decrease in the intermediation cost. Built upon Angeletos (2007), the model features an incomplete market with idiosyncratic investment risk. The model structure is similar to the two-period model, except that the time-horizon is infinite and the economy is productive. Because of this, leveraged and non-leveraged household need to be motivated in a different way. In the dynamic model, this is done by assuming that the former has a higher expected return on investment.

4.1 The Model Environment

Time is discrete, indexed by \( t \in \{1, 2, \ldots, \infty\} \). There is a continuum of infinitely-lived households indexed by \( i \in [0, 1] \). At each period, each household is endowed with one unit of labor that is supplied inelastically to the labor market. The economy produces one homogeneous and perishable good. The household \( i \)'s consumption at time \( t \) is denoted by \( c_i^t \).

There are two types of household in the economy: H-type and L-type. The two types have different productivities in risky investment. \( \tau_i^t \in \{H, L\} \) denotes the household's type in period \( t \). The type is assumed to be Markovian, with Poisson switching rates of \( \lambda_{LH} \) (from L-type to H-type) and \( \lambda_{HL} \) (from H-type to L-type). By the law of large numbers, in each period, there is always a fraction of \( p \) H-type agents, and a fraction of \( (1 - p) \) L-type agents,

\[
p = \frac{\lambda_{LH}}{\lambda_{HL} + \lambda_{LH}}. \tag{4.1}
\]

The household can invest in two assets: (1) risky investment in production \( i_i^t \geq 0 \); and (2) risk-free savings \( s_i^t \geq 0 \) in the financial intermediary. Households can also borrow from the financial intermediary to finance her risky investment. In this case, \( s_i^t < 0 \), and she has to pay a borrowing interest rate \( \pi_i^t \) which depends on her type and her leverage.

4.2 Preferences

The household’s preferences take the form of the Epstein-Zin specification\(^{18}\) with a constant elasticity of intertemporal substitution (EIS) and a constant relative risk aversion. A stochastic consumption stream \( \{c_i^t\}_{t=0}^{\infty} \) generates a stochastic utility stream \( \{u_i^t\}_{t=0}^{\infty} \) according to the following recursive form:

\[
U_i^t = \left[ (1 - \beta)(c_i^j)^{1-\mu} + \beta \left( E_t(U_{i+1}^{t+1})^{1-\gamma} \right)^{\frac{1-\mu}{1-\gamma}} \right]^{\frac{1}{1-\mu}}, \tag{4.2}
\]

where \( \mu > 0 \) measures the inverse of the EIS, and \( \gamma > 0 \) is coefficient of relative risk aversion. For \( \mu = 1/\gamma \), equation (4.2) reduces to the standard sum of expected CRRA instantaneous utility. Epstein-Zin preferences allows us to generate an empirically reasonable risk premium. However, none of the conclusions of this paper rely on the Epstein-Zin specification.

### 4.3 Investment and Production

Similar to the two-period model, the risky investment in production takes two stages. The first stage happens at the end of period \( t \), when the household invests the amount of \( i_t^i \) into her own capital-formation project. The project transforms investment into productive capital of the next period according to

\[
k_{i+1}^i = A^i_t(i_t^i)\nu.
\]

(4.3)

\( \nu \in (0, 1) \) represents the Lucas’ span of control, and \( \nu < 1 \) is necessary to prevent the household from accumulating wealth infinitely. \( A^i_t \) is the productivity of the project, which takes the form of

\[
\log A^i_t = \eta^i_t + \varepsilon^i_t, \quad \varepsilon^i_t \sim \mathcal{N}(-\sigma^2/2, \sigma^2),
\]

(4.4)

where \( \varepsilon^i_t \) is an exogenously normally distributed random variable that represents the risk of investment. Note that the expectation of \( \varepsilon^i_t \) is set to be \(-\sigma^2/2\) to normalize \( \mathbb{E}(A^i_t) = \exp\{\eta^i_t\} \). \( \eta^i_t \) differentiates the H- and L-type. We let \( \eta^H_t = \bar{\eta} > 0 \) for the H-type (\( \tau^H_t = H \)), and \( \eta^L_t = 0 \) for the L-type (\( \tau^L_t = L \)). In other words, the H-type has higher productivity in capital formation than the L-type. The differentiation between types of household based on their productivity is commonly done in the macroeconomic literature, such as Angeletos (2007) and Cao and Luo (2017).

The second stage takes place at the beginning of the next period, when the formed capital is pooled by a perfectly-competitive representative corporation. The corporation uses aggregate capital \( K_{t+1} \) and aggregate labor \( L_{t+1} \), and produces the period-(\( t+1 \)) consumption good \( Y_{t+1} \) using a Cobb-Douglas production technology:

\[
Y_{t+1} = K_{t+1}^\alpha L_{t+1}^{1-\alpha},
\]

(4.5)

where \( L_{t+1} \equiv 1 \), since we have normalized the inelastic labor supply to 1; and

\[
K_{t+1} = \int_0^1 k_{i+1}^i \, di.
\]

(4.6)

The corporation solves the static optimization problem

\[
\max_{\{K_{t+1}, L_{t+1}\}} \left\{ Y_{t+1} - w_{t+1} L_{t+1} - (r_{t+1}^c + \delta)K_{t+1} \right\},
\]

(4.7)
where \( w_{t+1} \) and \( r^c_{t+1} \) are the wage and the return on capital, respectively, and \( \delta \) is the depreciation rate of capital. The price of the consumption good is normalized to 1. The corporation is price-taker on the labor and capital markets. Solving the optimization problem above obtains

\[
w_{t+1} = (1 - \alpha)K^\alpha_{t+1}, \tag{4.8}
\]
\[
r^c_{t+1} = \alpha K^{\alpha - 1}_{t+1} - \delta. \tag{4.9}
\]

The investment-production specification resembles the formulation of the noncorporate and corporate sector by Quadrini (2000). However, in Quadrini (2000), the two sectors are parallel, while in our model, the two processes are sequential. The purpose of our set-up of investment and production is to decompose the investment returns. Specifically, the return on investment is determined by three components: the risk component, the type component (the ability or productivity of the investor), and the scarcity component. In the model, the risk component is captured by the random noise of \( \varepsilon^i_t \). The scarcity of capital is reflected by the capital return \( r^c_t \). Because of the resource constraints, especially the constraint of the total labor force, capital has diminishing marginal returns. Finally, the H-type realizes on-average higher return on investment, which is captured by the type-variable \( \eta^i_t \). Therefore, with our set-up of investment and production, different components of the investment returns can be analyzed separately.

### 4.4 Saving and Borrowing

Saving and borrowing are conducted through a perfectly-competitive and risk-neutral financial intermediary.\(^\text{19}\) Both saving and borrowing have a time duration of one period. Saving is always risk-free with an interest rate of \( r^f_t \), where \( r^f_t \) is determined by the market-clearing condition of the credit market. The interest rate of borrowing \( \pi^i_t \) is determined endogeneously by the bank, so that the bank makes zero expected profits.

It is assumed that the bank knows the type of the household when it makes its lending decision. Similar to the two-period model, borrower has limited liability, in the sense that the agent’s borrowing is only secured by her investment.\(^\text{20}\) The labor income, on the other hand, cannot be used for debt repayment. Suppose that a household borrows \( s^i_t < 0 \) to finance part of her investment, and her investment forms the capital \( k^i_{t+1} \). Next period, the household’s payable to the bank is

\[
u^i_{t+1} := -(1 + \pi^i_t)s^i_t. \tag{4.10}
\]

\(^\text{19}\) We can imagine that there are many identical risk-neutral banks operating in the credit market.

\(^\text{20}\) Similar to the assumption in the two-period model, one can imagine that each agent invests by setting up a limited-liability company. Even in the worst scenario, the bank can only grab the company’s asset.
However, the maximal commitment of the household to the borrowing is
\[ v_{t+1}^i := (1 - \varphi)(1 + r_{t+1}^c)k_{t+1}^i. \] (4.11)

If \( u_{t+1}^i > v_{t+1}^i \), the household losses all her capital and returns; and the bank can only collect \( v_{t+1}^i \).

An interpretation of this assumption is that when \( u_{t+1}^i \) is larger than \( v_{t+1}^i \), household would be forced to liquidate her capital, and the amount of \( \varphi(1 + r_{t+1}^c)k_{t+1}^i \) would be wasted on the lawsuit and the liquidation process. In this sense, \( \varphi \in (0, 1) \) is a measure of the bankruptcy deadweight loss. \( ^{21} \)

To summarize, let \( h_{t+1}^i \) be the household’s resource outflow due to borrowing \( s_t < 0 \),
\[
h_{t+1}^i = \begin{cases} 
- (1 + \pi_t^i)s_t^i, & \text{if } (1 + \pi_t^i)s_t^i \leq (1 - \varphi)(1 + r_{t+1}^c)k_{t+1}^i; \\
(1 + r_{t+1}^c)k_{t+1}^i, & \text{otherwise.}
\end{cases}
\] (4.12)

At the same time, define \( b_{t+1}^i \) as the debt repayment that the bank receives,
\[
b_{t+1}^i = \min \left\{ - (1 + \pi_t^i)s_t^i, (1 - \varphi)(1 + r_{t+1}^c)k_{t+1}^i \right\}.
\] (4.13)

Borrowing incurs an intermediation cost, while saving is cost-free. We assume that the intermediation cost is borne by the borrower, and is proportional to the amount of borrowing. Define \( L_t^i := - \frac{s_t^i}{(1 - \varphi)(1 + r_{t+1}^c)\exp\{\eta_t^i\}(i_t^i)^\nu} \) as a measure of leverage of the investment, the intermediation cost \( \kappa_t^i \) writes
\[
\kappa_t^i = - \theta(L_t^i)^\zeta s_t^i, \quad s_t^i < 0, \quad L_t^i > 0.
\] (4.14)

The term \((L_t^i)^\zeta\) is a technical term, with \( \zeta \) being a very small positive number. The existence of \((L_t^i)^\zeta\) doesn’t affect our quantitative results, but it makes the household’s next-period state a continuously differentiable function of the current choice variables, which facilitates our numerical solution to the model (see discussion below). The parameter \( \theta > 0 \) measures the level of the intermediation cost.

4.5 Timing of Events

The timing of events of the model is summarized in Figure 4.1. The household \( i \) enters period \( t \) with capital \( k_t^i \) and the saving (or borrowing) from the last period \( s_{t-1}^i \). The representative corporation

\( ^{21} \) A more realistic assumption would be that the household liquidates when \( u_{t+1}^i > (1 - \varphi^a)(1 + r_{t+1}^c)k_{t+1}^i =: \hat{v}_{t+1}^i \); and when liquidates, the amount of \( \varphi^b(1 + r_{t+1}^c)k_{t+1}^i \) is wasted, where \( 0 < \varphi^b < \varphi^a < 1 \). Under this assumption, when \( u_{t+1}^i > \hat{v}_{t+1}^i \), it would be optimal for the bank to “force” the household to liquidate. Here, \( \varphi^b \) would be the deadweight loss of bankruptcy. However, since bankruptcy is not a focus of this paper, we would like to make a little sacrifice on this part for the simplicity of the model by assuming that \( \varphi^a = \varphi^b =: \varphi \). Under this simplified assumption, the bank is indifferent to the liquidation of the household when \( u_{t+1}^i > v_{t+1}^i \). The liquidation is in this case only mechanical.
produces at the beginning of period $t$. The wage $w_t$ as well as the capital return $r^c_t$ are realized. At this point of time, we can define “cash-on-hand” $a^i_t$, which refers to the resources that the household $i$ possesses before making current-period consumption, investment and saving/borrowing decisions. Namely,

$$a^i_t = \begin{cases} 
  w_t + (1 + r^c_t)k^i_t + (1 + r^f_t)s^i_{t-1}, & \text{if } s^i_{t+1} \geq 0; \\
  w_t + (1 + r^c_t)k^i_t - h^i_t, & \text{if } s^i_{t+1} < 0.
\end{cases} \tag{4.15}$$

The household makes her decisions on consumption, investment and borrowing, before she learns her current-period type $\tau^i_t$, $\varepsilon^i_t$, however, is realized after investment has been made. At the end of period $t$, the next-period productive capital $k^i_{t+1}$ is formed.

Because of the existence of the intermediation cost, households do not simultaneously borrow and save. Define $n^i_t := a^i_t - c^i_t - i^i_t$ as the household’s residual resources after consumption and investment. If $n^i_t < 0$, she has to borrow $s^i_t = n^i_t - \kappa^i_t$. The intermediation cost is modelled as additional borrowing that has to be borne by the household, which is similar to the simplified model. For a more general representation of $s^i_t$ that covers both saving and borrowing, we define $\mathcal{I}^i_t$ as an indicator, which takes

**Figure 4.1.** Timing of events.
the value of 1 if \( n^i_t < 0 \) and 0 otherwise. \( s^i_t \) can be represented as

\[
s^i_t = a^i_t - c^i_t - i^i_t - \kappa^i_t I^i_t. \tag{4.16}
\]

Notice that \( s^i_t \geq 0 (< 0) \) if and only if \( n^i_t \geq 0 (< 0) \). Finally, in the model, it is assumed that when the bank makes lending decisions, it observes the household’s type \( \tau^i_t \), but does not know the realization of her \( \varepsilon^i_t \).

### 4.6 Borrowing Interest Rate

The bank pays the risk-free interest to the savers, and charges interest from the borrowers, and makes zero expected profit. For the household who borrows \( s^i_t < 0 \), the bank sets the borrowing interest rate \( \pi^i_t \) according to

\[
(1 + r^f_t) s^i_t + \int_{-\infty}^{\infty} b^i_{t+1} f(\varepsilon^i_t) d\varepsilon^i_t = 0. \tag{4.17}
\]

The first term is the cost to the bank, which is the gross risk-free interest paid by the bank to the savers. The second term is the expected gross lending interest received by the bank. Equation (4.17) defines a non-linear equation for \( \pi^i_t \) for any given triplet \( (s^i_t, i^i_t, \tau^i_t) \). Proposition 4.2 states that the borrowing interest rate \( \pi^i_t \) is finite when \( L^i_t < (1 + r^f_t)^{-1} \). As it turns out, \( \pi^i_t \) defined in (4.17) can be represented as a nonlinear equation with respect to \( L^i_t \), as is shown in the proposition below.

**Proposition 4.1** When \( i^i_t > 0 \), \( s^i_t < 0 \) and \( 0 < L^i_t < (1 + r^f_t)^{-1} \), \( \pi^i_t \) defined by the equation (4.17) is equivalent to the one defined by the following non-linear equation:

\[
1 + r^f_t = (1 + \pi^i_t) \left[ 1 - \Phi \left( \frac{\log \left( (1 + \pi^i_t)L^i_t \right) + \sigma^2/2}{\sigma} \right) \right] + (L^i_t)^{-1} \Phi \left( \frac{\log \left( (1 + \pi^i_t)L^i_t \right) - \sigma^2/2}{\sigma} \right), \tag{4.18}
\]

where \( \Phi(\bullet) \) is the cumulative density function (CDF) of the standard normal distribution.

**Proof** see appendix. \( \blacksquare \)

Given \( s^i_t, i^i_t \) and \( \tau^i_t \), Proposition 4.1 allows us to obtain \( \pi^i_t \) conveniently by solving the non-linear equation (4.18). This significantly simplifies our numerical solution to the model. To interpret equation (4.18), let’s re-arrange the equation into the following equivalent form:

\[
1 + r^f_t = (1 + \pi^i_t) \cdot \mathbb{P} \left[ \varepsilon^i_t \geq \log \left( (1 + \pi^i_t)L^i_t \right) \right] + (L^i_t)^{-1} \cdot \mathbb{P} \left[ \varepsilon^i_t < \log \left( (1 + \pi^i_t)L^i_t \right) - \sigma^2 \right]. \tag{4.19}
\]
The probability \( P[\varepsilon^t_i \geq \log((1 + \pi^t_i)L^t_i)] \) is the probability of full repayment;\(^{22}\) the probability \( P[\varepsilon^t_i < \log((1 + \pi^t_i)L^t_i) - \sigma^2] \) can be regarded as the equivalent probability of insolvency, and the term \((L^t_i)^{-1}\) can be deemed as the equivalent recovery rate when the borrower becomes insolvent.\(^{23}\) The lender chooses the interest rate \( \pi^t_i \) such that the expected repayment of one dollar of lending equals the risk-free (gross) return to the dollar.

It can be shown that \( \pi^t_i \) is a well-defined monotonically increasing function of \( L^t_i \). A unique finite solution to \( \pi^t_i \) is guaranteed for any \( L^t_i \in (0, (1 + r^f_t)^{-1}) \).

**Proposition 4.2** When \( i^t_i > 0 \) and \( s^t_i < 0 \), \( \pi^t_i \) is a well-defined function of \( L^t_i \). We represent the function as \( \pi^t_i = \Pi(L^t_i) : L^t \rightarrow R^t \), where \( L^t = (0, (1 + r^f_t)^{-1}) \) and \( R^t = (r^f_t, \infty) \). Further, when \( L^t_i \in L_t \), the first-order derivative of \( \pi^t_i \) with respect to \( L^t_i \) is

\[
\frac{d\pi^t_i}{dL^t_i} = (L^t_i)^{-2} \Phi\left(\frac{\log((1 + \pi^t_i)L^t_i) - \sigma^2/2}{\sigma}\right) \frac{1 - \Phi\left(\frac{\log((1 + \pi^t_i)L^t_i) + \sigma^2/2}{\sigma}\right)}{1 - \Phi\left(\frac{\log((1 + \pi^t_i)L^t_i) - \sigma^2/2}{\sigma}\right)} > 0. \tag{4.20}
\]

**Proof** see appendix. \( \blacksquare \)

Proposition 4.2 provides the upper bound of leverage \( L^t_i \) such that \( \pi^t_i \) is finite. This upper bound implicitly defines the borrowing limit of the household. Take the definition of \( L^t_i \) to the inequality \( L^t_i < (1 + r^f_t)^{-1} \), we have

\[
s^t_i \geq -\frac{(1 - \varphi)(1 + r^f_{t+1})\exp\{\eta^t_i\}}{1 + r^f_t} \cdot (i^t_i)^\nu. \tag{4.21}
\]

Depending on values of parameters and variables, the borrowing limit can be very low; But because of the decreasing returns to scale in investment \( (\nu < 1) \), such borrowing limit always exists. To prevent the borrowing limit to be too low, we require that in the stationary equilibrium,

\[
\frac{(1 - \varphi)(1 + r^f_{t+1})\exp\{\bar{\eta}\}}{1 + r^f_t} < 1. \tag{4.22}
\]

The assumption (4.22) is a restriction on the value of the parameter \( \varphi \). This restriction is meant to prevent even the H-type from investing purely by borrowing when \( i^t_i \geq 1 \). But because \( \nu < 1 \), investing purely by borrowing is always possible for either type when \( i^t_i \) is very small, as long as \( \varphi < 1 \). As it turns out, the condition (4.22) puts a very loose restriction on \( \varphi \). In our calibration, a \( \varphi \geq 0.1 \) guarantees that condition (4.22) is fulfilled.

\(^{22}\) See the proof to the Proposition 4.1 in the Appendix C.1.

\(^{23}\) “Equivalent” in the sense of transforming the continuous problem to the binary counterpart.
4.7 Stationary Recursive Competitive Equilibrium

The household has choices over consumption $c^t_i$, investment $i^t_i$ and saving/borrowing $s^t_i$. But by equation (4.16), for any $a^t_i$, once the $c^t_i$ and $i^t_i$ is chosen, $s^t_i$ is determined mechanically\(^{24}\). Therefore, we can reduce the household’s choice variables to $c^t_i$ and $i^t_i$. Moreover, in a given period, the aggregate state of the economy is characterized by the risk-free interest rate $r^t_i$ of the current period, as well as the aggregate capital $K_{t+1}$, capital return $r^c_{t+1}$ and wage $w_{t+1}$ of the next period. We let $G_t$ denote the period $t$’s aggregate state vector $(r^t_i, K_{t+1}, r^c_{t+1}, w_{t+1})$. Given $G_t$ and the household’s state $a^t_i$ and $\tau^t_i$, the household’s next-period cash-on-hand can be represented as a function of her choices $c^t_i, i^t_i$ and of random realization of $\varepsilon^t_i$\(^{25}\).

\[
a^t_{i+1} = \begin{cases} w_{t+1} + (1 + r^c_{t+1}) \exp\{\eta^t_i + \varepsilon^t_i\}(\pi^t_i) + (1 + r^t_i)(a^t_i - c^t_i - i^t_i), & \text{if } s^t_i \geq 0; \\
 w_{t+1} + (1 + r^c_{t+1}) \exp\{\eta^t_i + \varepsilon^t_i\}(\pi^t_i) + (1 + \pi^t_i)(a^t_i - c^t_i - i^t_i - \kappa^t_i), & \text{if } s^t_i < 0 \text{ and solvent}; \\
 w_{t+1}, & \text{if } s^t_i < 0 \text{ and insolvent}. \end{cases} \tag{4.23}
\]

Next, we formulate the household’s problem recursively. Following the convention of the literature, we drop the subscript $t$, and we use the prime-notation to represent the variables in the period $(t+1)$. The recursive problem of the household writes

\[
\mathbb{V}(a^i, \tau^i; G) = \max_{(c^i, i^i) \in \mathcal{D}_+} \left[ (1-\beta)(c^i)^{1-\mu} + \beta \left( \lambda_H \int_{\varepsilon^i} \mathbb{V}\left( (a^i)', H; G' \right)^{1-\gamma} f(\varepsilon^i) \, d\varepsilon^i + \lambda_L \int_{\varepsilon^i} \mathbb{V}\left( (a^i)', L; G' \right)^{1-\gamma} f(\varepsilon^i) \, d\varepsilon^i \right)^\frac{1-\gamma}{1-\mu} \right], \tag{4.24}
\]

subject to (4.23),

where $\mathcal{D}_+$ is the range of $(c^i, i^i)$ with $c^i > 0$ and $i^i > 0$ that fulfill inequality (4.21), ensuring a finite interest rate. The household forms rational expectation on the next period’s aggregate state $G'$ when

\(^{24}\) It can be shown that given $G_t$, $a^t_i$ and $\tau^t_i$, both $L^t_i$ and $\kappa^t_i$ are well-defined monotonic functions of $(c^t_i, i^t_i)$. See Appendix D.1.

\(^{25}\) From equation (D.17), one can see how the inclusion of the technical term $(L^t_i)^\zeta$ in the formulation of the intermediate cost (4.14) “smooth out” the next-period state. With $(L^t_i)^\zeta$ in $\kappa^t_i$, $a^t_{i+1}$ is continuously differentiable in $c^t_i$ and $i^t_i$, especially when $(c^t_i + i^t_i)$ crosses the threshold of $a^t_i$. This is because when $(c^t_i + i^t_i) \uparrow a^t_i, (L^t_i)^\zeta \to 0$ and hence $\kappa^t_i \to 0$. The continuously differentiable state space allows us to explicitly formulate the household’s first-order conditions, which facilitates our numerical solution to the model. For a rigorous discussion of the continuous-differentiability of the state space, as well as the first-order conditions of the household’s optimization problem, please refer to Appendix D.
solving the individual optimization problem (4.24). The individual optimal decisions give rise to the distribution of saving/borrowing and future capital over households. These distributions, along with the type distribution over households, in turn determine the aggregate state $G'$, on which the individual household’s optimal decision is based. In the recursive equilibrium, the realization of the the aggregate state $G'$ is consistent with the household’s rational expectation.

**Definition 4.1 (Stationary Recursive Competitive Equilibrium)** A recursive competitive equilibrium is defined as: (1) The household’s individual value function $V(a^i, \tau^i; G)$; (2) The household’s individual decision rules $c^i = C(a^i, \tau^i; G)$ and $i^i = I(a^i, \tau^i; G)$ for consumption and investment, respectively; (3) Law of motion of the aggregate state $G' = g(G)$, such that

1. Given the rational expectation of $G'$, the household’s individual value function and decision rules solve the optimization problem (4.24);

2. Given the household’s individual decision rules, $G' = g(G)$ is determined as follows:
   (1) $K'$ is the sum of individual capital holdings, equation (4.6);
   (2) $K'$ determines $w'$ and $(r_c)'$ according to equations (4.8) and (4.9);
   (3) The risk free interest rate $r^f$ equates aggregate saving and borrowing,
   \[
   \int_0^1 s^i \, di = 0; \tag{4.25}
   \]

3. The next period’s aggregate state $G'$ determined in 2. is consistent with the household’s rational expectation in 1.

A stationary recursive competitive equilibrium is the recursive competitive equilibrium with invariant law of motion $g(G) = G$. 

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5 Calibration

As a benchmark, the dynamic model is calibrated annually to the US economy from 1980 to 2007. The logic for the choice of such time window follows the empirical part. In the model, there are 13 parameters to be calibrated: \((\beta, \mu, \gamma, \sigma, \theta, \xi, \phi, \chi, \lambda_{HL}, \lambda_{LH}, \alpha, \delta)\). Some parameters are calibrated outside the model, and others are calibrated inside. Table 5.1 summarizes the benchmark calibration.

Table 5.1. Benchmark calibration.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Target/Source</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Calibrated Outside the Model</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.96</td>
<td>(conventional)</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>0.36</td>
<td>(conventional)</td>
</tr>
<tr>
<td>(\delta)</td>
<td>0.08</td>
<td>(conventional)</td>
</tr>
<tr>
<td>(\lambda_{HL})</td>
<td>0.10</td>
<td>Cao and Luo (2017)</td>
</tr>
<tr>
<td>(\lambda_{LH})</td>
<td>0.0111</td>
<td>10% share of H-types among household</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.20</td>
<td>Angeletos (2007), Benhabib et al. (2011), etc.</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>(10^{-4})</td>
<td>(very small positive number)</td>
</tr>
<tr>
<td><strong>Calibrated Inside the Model</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\mu)</td>
<td>0.50</td>
<td>risk-free rate</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>1.50</td>
<td>risk premium</td>
</tr>
<tr>
<td>(\theta)</td>
<td>0.06</td>
<td>top-10% wealth share</td>
</tr>
<tr>
<td>(\nu)</td>
<td>0.9945</td>
<td>top-1% wealth share</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.025</td>
<td>corporate debt-to-GDP</td>
</tr>
<tr>
<td>(\varphi)</td>
<td>0.10</td>
<td>income Gini-coefficient</td>
</tr>
</tbody>
</table>
5.1 Externally Calibrated Parameters

The “H-type” is interpreted as the best 10% investors (households). In other words, the unconditional proportion of the H-type is $p = 0.10$. Following Cao and Luo (2017), we set $\lambda_{HL} = 0.10$ such that the average duration of being H-type is 10 years. By the property of the Markov chain, $p := \frac{\lambda_{HL}}{\lambda_{HL} + \lambda_{LH}} = 0.10$, this implies that $\lambda_{LH} = \frac{p}{1-p} \lambda_{HL} = 0.0111$.

Two sources of data can be used to calibrate the risk of investment (the volatility of the capital formation) $\sigma$. The first is the volatility of returns on public equity. Between 1980 and 2007, the annual real returns of the S&P 500 have a volatility of 17.52%. However, in our model, capital investment goes beyond public equity. Regarding private equity, Moskowitz and Vissing-Jørgensen (2002) find that private equity does not offer higher returns than public equity, but the owners of the private equity (entrepreneurs) poorly diversify their portfolio. The concentrated investment on private equity implies a higher risk. However, there is no reliable measure on such risk. In his baseline calibration, Angeletos (2007) calibrates the volatility to the entrepreneurial returns to be 20%. Benhabib et al. (2011) adopt the number of 20% to be their calibration for the volatility of overall investment (namely, public and private equity combined). We follow Angeletos (2007) and Benhabib et al. (2011) and calibrate $\sigma$ to 20%.

The rest of the parameters are calibrated as follows. The subjective discount factor $\beta$, the share of capital income in production $\alpha$, and the annual capital depreciation rate $\delta$ are set to conventional value of 0.96, 0.36 and 0.08, respectively. These values are standard in the literature, and are identical to the calibration in Aiyagari (1994) and Angeletos (2007). The very small but positive technical term $\zeta$ is set to be $1 \times 10^{-4}$.

5.2 Internally Calibrated Parameters

The remaining six parameters are calibrated internally: $(\mu, \gamma, \eta, \nu, \theta, \varphi)$. $\mu$ is the inverse of the elasticity of intertemporal substitution (EIS). The empirical estimations to the EIS differ wildly, ranging from 0.1 (Hall, 1988) to 2 (Gruber, 2013). Calibrated quantitative works therefore use

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26 The value of $p$ is chosen rather arbitrarily. $p$ is linked to the H-type’s return premium on risky investment $\eta$. Since $\eta$ is calibrated inside the model, if one uses a different value of $p$, a corresponding change to the value of $\eta$ must be made. For example, if one chooses $p = 0.01$ so that the H-type has a proportion of 1% among all households, the H-type’s return premium $\eta$ must be higher so that calibration targets can be matched. As a matter of fact, the conclusions of our model won’t change if one uses different values of $p$.

Table 5.2. Moments of the benchmark calibration.

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Data</th>
<th>Model (benchmark)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Targeted Statistics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk-free rate $r^f$ (%)</td>
<td>2.03</td>
<td>2.05</td>
</tr>
<tr>
<td>Average return on investment (ROI, %)</td>
<td>8.33</td>
<td>8.65</td>
</tr>
<tr>
<td>Average risk premium (%)</td>
<td>6.30</td>
<td>6.60</td>
</tr>
<tr>
<td>(Corporate) Debt-to-GDP</td>
<td>0.59</td>
<td>0.57</td>
</tr>
<tr>
<td>Top-10% wealth share</td>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>Top-1% wealth share</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>Income Gini-coefficient</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td><strong>Untargeted Statistics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Capital-output ratio</td>
<td>2.71</td>
<td>2.71</td>
</tr>
<tr>
<td>Wealth Gini-coefficient</td>
<td>0.79</td>
<td>0.69</td>
</tr>
</tbody>
</table>

widely different values of the EIS. For example, Barro (2009) and Colacito and Croce (2011) calibrate the EIS to 2; Angeletos (2007) applies the value of 1 in his benchmark calibration; and in Guvenen (2009) the calibrated EIS is 0.1–0.3. In this paper, $\mu$ is calibrated internally to match the real risk-free rate of 2.03%.

The range of estimated values of the relative risk aversion $\gamma$ is also wide. The estimates for $\gamma$ range from 0-1 (Hansen and Singleton, 1982, 1984) to 40-50 (Cochrane and Hansen, 1992). Estimates based on labor-market data usually imply a lower upper-bound of the parameter. For example, Chetty (2006) shows that the empirical evidence on the effects of wage changes to labor supply imply that $\gamma < 2$. On the other hand, empirical work using data of the financial market leads to higher estimates, such as Bliss and Panigirtzoglou (2005), who estimate $\gamma$ to be between 3.37 and 9.52, depending on the forecast horizon. Given such large variety of empirical estimates of $\gamma$, I choose to calibrate the parameter internally. The $\gamma$ is calibrated to target the average risk premium of the risky asset of 6.30% during 1980 and 2007.
The parameters of $\eta$ (return premium of the H-type) and the $\nu$ (the Lucas’ span-of-control) are calibrated to target the wealth distribution. To be specific, $\eta$ is set to target the top 10% wealth share in the US, which averages 65% during 1980 and 2007; and $\nu$ is calibrated to target the top 1% wealth share of 29% during the same period. The bankruptcy deadweight loss $\varphi$ affects household’s willingness and ability to leverage. We will see in the next section that, in the model, leverage exaggerates the capital-income risk, which drives the income inequality. In this sense, we calibrate $\varphi$ to the average income Gini-coefficient of 0.48.

There is no direct indicator in the data that leads to a precise calibration of the intermediation cost of borrowing $\theta$. To obtain an idea about the range of $\theta$, we calculate the average prime-lending spread in the United States. The average of such spread between 1980 and 2007 is 2.75%. However, even the prime borrowers bear some default risk. Therefore, $\theta$ should be slightly lower than 2.75%. In our model, we choose the value of $\theta = 2.50\%$ to match the average debt-to-GDP ratio of the non-financial corporate sector of 59%.

Table 5.2 compares some key data moments to the moments generated by our model. In general, we match our calibration targets well. Moreover, the capital-output ratio of the benchmark model is also in line with the data, although we don’t target this statistic. However, the wealth Gini-coefficient generated by the model is lower than what the data suggest, because our model precludes negative wealth.

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28 To obtain the capital-output ratio in the data, we use the total tangible fixed assets as our measure of capital stock. See BEA Fixed Assets Table 1.1.
6 Results

6.1 Household’s Decisions

Individual household’s policies of investment and saving as functions of cash-on-hand are reported in Figure 6.1. The baseline considers $\theta = 2.5\%$. To compare, we include the counterfactual policies of the household when $\theta$ is 2% and 3%.

In any state, the H-type invests more than the L-type; and consumes less than the L-type. In the model, the rich L-type household saves; while the H-type only saves when her cash-on-hand is

**Figure 6.1.** Policies of the individual household.

![Figure 6.1](image)

**Notes:** The dotted curves are policies for $\theta = 2\%$; the solid curves are policies when $\theta = 2.5\%$; and the dashed curves represent the policies when $\theta = 3\%$. 

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very large.\textsuperscript{29} The savings of the rich L-types finance the borrowings of the H-types. But interestingly, when \( a^i \) is small enough, L-type also borrows to invest. This is because the lack of labor-income risk provides a consumption insurance to the household. In the model, household’s cash-on-hand cannot drop below the stationary wage \( w \). The lower bound on the cash-on-hand limits the marginal cost of insolvency; but because of the concavity of the preferences, the marginal benefit of leveraged investment gets higher when household becomes poorer. Thus the poor L-type also leverages.\textsuperscript{30} On the other hand, the H-type de-leverages when she becomes richer. As a result, her borrowing interest rate also decreases. The reason behind is straightforward. With the decreasing marginal utility from future consumption, the subjective marginal return from the leveraged investment drops when \( a^i \) increases. Hence leverage becomes less attractive to the richer H-type.

In Figure 6.1, the counterfactual policies when \( \theta = 3\% \) and 2\% are plotted along with the policies under the benchmark calibration. When it comes to the counterfactuals of consumption-, saving- and leverage-policies, the dynamic model has similar implications as the two-period model. The decreasing intermediation cost encourages more leveraged investment of the H-type. The interest rate of the H-type’s borrowing also rises, due to the increasing leverage. Later in Subsection 6.2 we will show that, in general equilibrium, lower \( \theta \) results in the decrease in the capital return \( r^c \) and the increase in the risk-free rate \( r^f \) (see Table 6.1). On the individual level, the lower \( r^c \) and higher \( r^f \) discourages the (not-too-poor) L-type from investing, and encourages the rich L-type to save.\textsuperscript{31}

In Figure 6.2, we highlight the episodes of the value and policy functions when \( a \) is small and when \( a \) is large. First, the values of the household shown on the top-left panel indicate that the reduction in \( \theta \) is beneficial to all household, except for the rich H-type. The latter group of household suffers from the decrease in \( r^c \); while with the accumulation of wealth, the benefit of easier credit tempers with reduced motivation to leverage. On the other hand, the rich L-type still benefits from the reduction in \( \theta \), because the saving’s interest rate \( r^f \) rises and the rich L-type allocates significant proportion of her wealth in the risk-free savings. Second, decreasing \( \theta \) encourages the poor H-type to reduce consumption; but for other households, the lower \( \theta \) is associated with higher consumption

\textsuperscript{29} With our baseline calibration (\( \theta = 2.5\% \)), H-type starts to save when \( a^i > 5000 \). In the stationary equilibrium, the probability of any household accumulating such large \( a^i \) is negligible.

\textsuperscript{30} Although the poor L-type borrows, the borrowing amount is rather small compared with the H-type, both at the individual level (as can be see from the corresponding graphs in the Figure 6.1 and Figure 6.2) and in aggregate. The high leverage of the very poor L-type shown in the bottom-left panel of the Figure 6.1 is due to the very small amount of investment. In aggregate, the borrowing of the poor L-type represents no more than 5\% of the total borrowing.

\textsuperscript{31} For the poor L-type, the reduction in \( \theta \) encourages her to leverage and to invest more. See Figure 6.2. This is because lower \( \theta \) further increases the marginal benefit of leveraged investment.
Figure 6.2. Values and policies of the individual household in detail

(see the bottom-left panel of Figure 6.2). This implies that the substitution effect brought by lower $\theta$ dominates its income effect only for the poor H-type. Last but not least, we look at the investment and saving/borrowing policies plotted on the right two panels of the same figure. Here, the decrease in $\theta$ encourages the H-type to leverage more to invest. However, as has been discussed above, when the L-type is poor, she also leverages. With lower $\theta$, poor L-type also leverages more, although the amount of borrowing by the poor L-type is very small. The rich L-type, on the other hand, rebalances her portfolio from the risky investment toward safe savings, when she faces lower $\theta$. 

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In summary, lower intermediation cost encourages the poor H-type to reduce consumption and increase leverage to invest more. When the H-type gets richer, the income effect of lower $\theta$ starts to dominate its substitution effect, making her consume more than when $\theta$ is higher. But in any state, the H-type’s leverage and investment are higher with lower $\theta$. The reduction in $\theta$ improves the subjective value of the poor H-type, but depresses the value of the rich H-type. The rich L-type, at the same time, rebalances her portfolio toward risk-free savings. Her consumption is higher and the subjective value is improved with lower $\theta$.

6.2 General Equilibrium

In Table 6.1, we summarize the effect of $\theta$ on the general equilibrium. As a counterfactual experiment, we allow $\theta$ to vary from 3.50% to 1.50%, while keeping the values of other parameters at the benchmark. When $\theta$ decreases, the credit-output ratio rises substantially, while the change in the capital-output ratio is less significant. This can be explained as follows. The decrease in $\theta$ propels the H-type to leverage more to invest, which drives up the aggregate capital stock $K$ and lowers capital return $r^c$. The fall in $r^c$ induces the rich L-type to reduce her investment in risky capital and put more resources into savings. Therefore, the credit market sees a “simultaneous” credit expansion from both supply side (rich L-types) and demand side (H-types), which drives up the credit-output ratio substantially. In the capital market, the additional investment by the H-type is dampened by the reduction in investment by the rich L-type. This explains why the capital expansion is moderate compared to the credit expansion.

Since the additional demand for credit by the H-type is largely met by the additional supply by the rich L-type, the price of the credit $r^f$ does not change much. One sees from Table 6.1 that when $\theta$ decreases, $r^f$ increases quite moderately. Similar to the two-period model, the H-type is higher motivated to invest than the L-type. Namely, if each is given one unit of additional resource, the H-type would distribute larger proportion of the additional dollar into investment than the L-type. In this case, the response of the H-type dominates the response of the L-type. In the credit market, it implies that the demand-effect (by H-type) surpasses the supply-effect (by rich L-type). Hence we see an increase in the risk-free rate (however moderately) when $\theta$ decreases. In the capital market, the additional investment by the H-type dominates the reduction in investment by the L-type, implying larger capital-output ratio and lower $r^c$.

We now look at the capital market in specific. Remember that the model’s set-up on the invest-

\footnote{The table includes the case of $\theta = 3.00\%$ and $\theta = 2.00\%$, which are cases for which we plot counterfactual policies in Figure 6.1, 6.2, 6.3 and 6.4.}
Table 6.1. Effects of *cateris paribus* change of $\theta$ on returns, as well as on aggregate capital and credit.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r^f$</th>
<th>$r^c$</th>
<th>avg.ROI</th>
<th>capital-output ($K/Y$)</th>
<th>credit-output</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.50</td>
<td>2.00</td>
<td>5.42</td>
<td>8.37</td>
<td>268.18 ($-1.32%$)</td>
<td>42.89 ($-25.44%$)</td>
</tr>
<tr>
<td>3.25</td>
<td>2.01</td>
<td>5.38</td>
<td>8.44</td>
<td>269.03 ($-1.02%$)</td>
<td>46.39 ($-19.36%$)</td>
</tr>
<tr>
<td>3.00</td>
<td>2.03</td>
<td>5.34</td>
<td>8.51</td>
<td>269.84 ($-0.72%$)</td>
<td>50.02 ($-13.05%$)</td>
</tr>
<tr>
<td>2.75</td>
<td>2.04</td>
<td>5.29</td>
<td>8.58</td>
<td>270.85 ($-0.34%$)</td>
<td>53.82 ($-6.45%$)</td>
</tr>
<tr>
<td>2.50</td>
<td>2.05</td>
<td>5.24</td>
<td>8.65</td>
<td>271.81</td>
<td>57.53</td>
</tr>
<tr>
<td>2.25</td>
<td>2.06</td>
<td>5.19</td>
<td>8.71</td>
<td>272.92 ($+0.40%$)</td>
<td>61.37 ($+6.67%$)</td>
</tr>
<tr>
<td>2.00</td>
<td>2.09</td>
<td>5.14</td>
<td>8.77</td>
<td>273.92 ($+0.77%$)</td>
<td>66.30 ($+15.24%$)</td>
</tr>
<tr>
<td>1.75</td>
<td>2.12</td>
<td>5.08</td>
<td>8.82</td>
<td>275.24 ($+1.26%$)</td>
<td>72.18 ($+25.47%$)</td>
</tr>
<tr>
<td>1.50</td>
<td>2.19</td>
<td>5.03</td>
<td>8.83</td>
<td>276.28 ($+1.64%$)</td>
<td>79.87 ($+38.83%$)</td>
</tr>
</tbody>
</table>

Notes: “avg.ROI” is the weighted average return on investment across households. The numbers in parentheses are percentage changes from benchmark calibration ($\theta = 2.5\%$). All other numbers are in percentage.

ment and production distinguishes three components of the return on investment (ROI): (1) the risk component, (2) the type component and (3) the scarcity component (measured by $r^c$). The larger capital-output ratio reduces scarcity of capital, hence $r^c$ drops. This means that investment becomes less rewarding for the L-type. The difference between the average ROI and $r^c$ measures the type component of the investment return. One can observe in Table 6.1 that the increasing leverage brought by lower $\theta$ exaggerates such type component. Regarding the risk component, while the exaggerated type component brings higher average returns to the H-type, it also introduces higher capital-income risk among H-types. This is due to two reasons. First, higher leverage translates to higher risk of insolvency; and second, leverage amplifies the profits and losses of investment. As will be discussed in the next subsection, the higher capital-income risk is the force that drives up the inequality when the $\theta$ decreases. For the L-types, however, the capital-income risk is subdued when $\theta$ is lower, because they allocate higher proportion of their wealth to risk-free savings.

---

33 Similar to the discussion in the two-period model, to be consistent with existing literature, we define the “capital income” as the total income from investment and risk-free savings.
6.3 Income and Wealth Distributions

Figure 6.3 plots the distribution of cash-on-hand in the stationary recursive competitive equilibrium. The distribution is skewed and has a heavy right-tail. Under the benchmark calibration, the top one-percent of household holds about 25% of the total cash-on-hand; while the top 10-percent household’s share of cash-on-hand is more than 60%. As in Benhabib et al. (2011, 2015), the skewness and the heavy tail of the wealth distribution is driven by the capital-income risk.

Figure 6.3 also plots the distributions when $\theta = 0.03$ and 0.02, respectively. On one hand, when $\theta$ decreases, the mode of the wealth distribution shifts to the left, implying a decrease in the median wealth of household. We further see in the left two panels of Figure 6.4 that the medians of the wealth distributions decrease for both of the subgroups of H- and L-types. The decrease in the median wealth is driven by the decrease in $r^c$ and the increase in risk brought by higher leverage. The reasons for the changes in medians are different for the two types. Remember that the L-type does not save when $a^i$ is low. Therefore, the poor L-type does not benefit from an increase in $r^f$, but is harmed by the

\textbf{Figure 6.3.} Distribution of cash-on-hand.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6_3.png}
\caption*{Notes: The two plots on the right highlight the distribution around the mode (top right) and when $a^i$ is large (bottom right).}
\end{figure}
falling $r^c$. Hence, when $\theta$ decreases, poor L-type’s wealth is shrinking, which drives down the median of the L-types’ wealth distribution. For the H-types, the higher leverage implies higher risk. Among other things, there are more insolvent H-types when $\theta$ decreases.\footnote{When $\theta$ drops from 0.03 to 0.02, the insolvency rate of the H-type (measured by the percentage-share of the H-types that become insolvent for the current period) increases from 2.31\% to 3.65\%.} This drives down the median of the wealth distribution among the H-types.

On the other hand, the right-tail of the distribution gets thicker if $\theta$ decreases (bottom-right panel of Figure 6.3). The pattern of the tail-change is the same for both of the subgroups of the H- and L-types, too, as is shown in the right two panels of the Figure 6.4. Again, the reasons behind the change in tail are different for the H- and L-types. The H-type leverages more when $\theta$ is lower. As is discussed above, higher leverage of the H-type exaggerates the type component of the investment returns and amplifies the capital-income risk. The H-types are thus able to accumulate more wealth on average, albeit with more heterogeneity. The increased heterogeneity in investment returns is
accountable for the increasing thickness of the right-tail. Turning to the L-types. Most of the very rich L-types accumulate wealth when they are H-types in the previous periods. Therefore, the thicker tail of the wealth distribution among the H-types leads to the thicker tail of that among the L-types, through the Markovian transition between types. Moreover, the increasing $r^f$ brought by the decreasing intermediation cost also helps to preserve the rich L-types’ wealth better.

Table 6.2. Effects of *ceteris paribus* change of $\theta$ on wealth and income distributions.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Gini</th>
<th>Gini L-type</th>
<th>top 10% share</th>
<th>top 1% share</th>
<th>Gini</th>
<th>Gini L-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.50</td>
<td>64.75</td>
<td>63.39</td>
<td>60.38</td>
<td>24.34</td>
<td>47.74</td>
<td>36.43</td>
</tr>
<tr>
<td>3.25</td>
<td>65.94</td>
<td>64.35</td>
<td>61.75</td>
<td>25.39</td>
<td>48.06</td>
<td>35.75</td>
</tr>
<tr>
<td>3.00</td>
<td>67.01</td>
<td>65.49</td>
<td>63.01</td>
<td>26.63</td>
<td>48.37</td>
<td>35.07</td>
</tr>
<tr>
<td>2.75</td>
<td>68.15</td>
<td>66.55</td>
<td>64.32</td>
<td>27.84</td>
<td>48.72</td>
<td>34.34</td>
</tr>
<tr>
<td><strong>2.50</strong></td>
<td><strong>69.41</strong></td>
<td><strong>67.76</strong></td>
<td><strong>65.67</strong></td>
<td><strong>29.08</strong></td>
<td><strong>49.08</strong></td>
<td><strong>33.62</strong></td>
</tr>
<tr>
<td>2.25</td>
<td>70.70</td>
<td>69.02</td>
<td>67.15</td>
<td>30.27</td>
<td>49.44</td>
<td>32.89</td>
</tr>
<tr>
<td>2.00</td>
<td>72.29</td>
<td>70.70</td>
<td>68.70</td>
<td>31.51</td>
<td>49.71</td>
<td>32.25</td>
</tr>
<tr>
<td>1.75</td>
<td>74.29</td>
<td>72.94</td>
<td>70.52</td>
<td>32.68</td>
<td>50.05</td>
<td>31.51</td>
</tr>
<tr>
<td>1.50</td>
<td>76.63</td>
<td>75.67</td>
<td>72.51</td>
<td>34.03</td>
<td>50.26</td>
<td>31.13</td>
</tr>
</tbody>
</table>

Notes: The household $i$’s wealth is defined as her cash-on-hand minus the stationary wage, namely, $a_i - w$. All numbers in the table are in percentage.

Table 6.2 documents different measures of wealth and income inequality for different values of $\theta$. The table shows that, in general, the decrease in the intermediation cost is associated with the rise of both income and wealth inequality. Such rise in inequality is associated with the change in the tail of the wealth distribution, as is evident from the change in the top 10% and 1% wealth share. However, we can see this by comparing the household’s expected next-period cash-on-hand $\mathbb{E}((a_i')')$ against the current-period state $a_i$. $\mathbb{E}((a_i')') > a_i$ implies that the household is accumulating wealth, otherwise she is reducing wealth. Remember that $\mathbb{E}((a_i')')$ is a monotonically increasing function of $a_i$. For $\tau^i = H, L$, we can solve for the equation $\mathbb{E}((a_i')') = a_i$. We denote the solution to the equation by $\bar{a}^H$ and $\bar{a}^L$ for the H- and L-type household, respectively. For the benchmark calibration ($\theta = 2.5\%$), $\bar{a}^L = 2.87$ while $\bar{a}^H > 5000$. Therefore, only through being H-types at some time, can a household becomes very rich.

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35 We can see this by comparing the household’s expected next-period cash-on-hand $\mathbb{E}((a_i')')$ against the current-period state $a_i$. $\mathbb{E}((a_i')') > a_i$ implies that the household is accumulating wealth, otherwise she is reducing wealth. Remember that $\mathbb{E}((a_i')')$ is a monotonically increasing function of $a_i$. For $\tau^i = H, L$, we can solve for the equation $\mathbb{E}((a_i')') = a_i$. We denote the solution to the equation by $\bar{a}^H$ and $\bar{a}^L$ for the H- and L-type household, respectively. For the benchmark calibration ($\theta = 2.5\%$), $\bar{a}^L = 2.87$ while $\bar{a}^H > 5000$. Therefore, only through being H-types at some time, can a household becomes very rich.
it is interesting to notice that the income-Gini among the L-type households decreases when $\theta$ drops. This observation is similar to what we have observed in the two-period model, and the explanation remains the same. The rich L-type, facing increasing $r^f$ and decreasing $r^c$, holds more risk-free savings and invests less. This essentially reduces the capital-income risk among the rich L-types, which reduces the income dispersion among them.

As a matter of fact, capital-income risk can be viewed as the main driving force behind the change in the income and wealth distributions. As mentioned earlier, when the intermediation cost decreases, the capital-income risk is exaggerated for the H-type and is subdued for the rich L-type. Higher capital-income risk translates to higher income dispersion. For the L-type, the income-Gini shrinks due to the reduced risk of capital income; while for the H-type, the opposite is happening. Larger income dispersion among H-types (as well as between the H- and L-type) dominates the concentration of income among L-types, resulting to a larger overall income inequality. As for the wealth, the story is the same for the H-type. But because wealth is an accumulation of previous income, the wealth inequality among L-types still increases due to L-types’ previous histories as the H-types, as well as a higher $r^f$ that better perserves the very rich L-types’ asset.

**Feedback to the Capital and Credit Markt.** There is interaction between the wealth distribution and the general equilibrium. The decrease in $\theta$ leads to a larger share of very rich L-type households (see the bottom-right panel of Figure 6.4). This increases the credit supply, because the rich L-types are savers, and the richer the L-type gets, the more she saves (see the top-right panel of Figure 6.1). Hence the thicker tail of L-types’ wealth distribution makes credit even more available to the H-type. This contributes to the reason why the $r^f$ does not increase much when $\theta$ drops. The additional credit availability promotes more borrowing and investment of the H-type, which again leads to larger dispersion of wealth. Therefore, this feedback between the wealth distribution and the general equilibrium exaggerates the previously-discussed feedback-loop between the credit market and capital market.

### 6.4 Welfare

We apply the welfare criterion à la Aiyagari and McGrattan (1998), where the welfare is defined as the weighted-average values of the households. Let $\Omega^H, \Omega^L$ represents the welfare of the group of the H-types and the L-types, respectively. In the stationary recursive competitive equilibrium, we define

$$
\Omega^H = \int_w^{\infty} V(a^i, H) \, dH(a^i, H), \quad \Omega^L = \int_w^{\infty} V(a^i, L) \, dH(a^i, L),
$$

(6.1)
where $H(a^i, H)$ and $H(a^i, L)$ are distributions (probability densities) of cash-on-hand for the H- and L-types, respectively. The overall welfare of the economy, $\Omega$, is defined as $\Omega = p\Omega^H + (1 - p)\Omega^L$. The welfare for other subgroups of households can be defined similarly.

**Table 6.3.** Welfare effects of *ceteris paribus* change of $\theta$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>overall</th>
<th>H-type</th>
<th>L-type</th>
<th>bottom 90%</th>
<th>top 1%-10%</th>
<th>top 1%</th>
<th>top 0.1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.50</td>
<td>-0.67</td>
<td>-2.63</td>
<td>-0.36</td>
<td>+0.14</td>
<td>-1.86</td>
<td>-12.70</td>
<td>-16.40</td>
</tr>
<tr>
<td>3.25</td>
<td>-0.52</td>
<td>-1.97</td>
<td>-0.28</td>
<td>+0.08</td>
<td>-1.21</td>
<td>-9.82</td>
<td>-13.45</td>
</tr>
<tr>
<td>3.00</td>
<td>-0.36</td>
<td>-1.42</td>
<td>-0.20</td>
<td>+0.04</td>
<td>-0.85</td>
<td>-6.57</td>
<td>-8.45</td>
</tr>
<tr>
<td>2.75</td>
<td>-0.19</td>
<td>-0.71</td>
<td>-0.10</td>
<td>+0.03</td>
<td>-0.52</td>
<td>-3.35</td>
<td>-3.73</td>
</tr>
<tr>
<td><strong>2.50</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.25</td>
<td>+0.22</td>
<td>+0.88</td>
<td>+0.11</td>
<td>-0.00</td>
<td>+0.51</td>
<td>+3.30</td>
<td>+4.10</td>
</tr>
<tr>
<td>2.00</td>
<td>+0.42</td>
<td>+1.64</td>
<td>+0.22</td>
<td>-0.01</td>
<td>+0.97</td>
<td>+6.87</td>
<td>+7.42</td>
</tr>
<tr>
<td>1.75</td>
<td>+0.66</td>
<td>+2.68</td>
<td>+0.35</td>
<td>-0.08</td>
<td>+2.03</td>
<td>+10.51</td>
<td>+11.06</td>
</tr>
<tr>
<td>1.50</td>
<td>+0.88</td>
<td>+3.39</td>
<td>+0.48</td>
<td>-0.18</td>
<td>+2.92</td>
<td>14.73</td>
<td>16.23</td>
</tr>
</tbody>
</table>

**Notes:** All numbers in the table are percentage change from the benchmark ($\theta = 2.5\%$).

Table 6.3 reports the change in welfare for the overall and selected subgroups of household, when $\theta$ changes. Similar to the two-period model, the decrease in the intermediation cost improves the overall welfare. However, in the dynamic model, the H-type’s overall welfare is also improving, and is improving faster than that of the subgroup of the L-types. This is different from the two-period model, since the H-type in the dynamic model is associated with higher ROI on average; while in the two-period model, the H-type make zero profits. Thus the H-type in the dynamic model benefits more from the decrease in $\theta$.

The dynamic model also allows us to assess the welfare effects for each wealth group. These are reported in the last four columns of the Table 6.3. As one can see, the decrease in $\theta$ reduces the welfare of the bottom-90% wealth group. For the top 10% wealth group, the reduction of the intermediation cost is welfare improving. The improvement of welfare is biased toward the rich household, as the percentage welfare gain of the top 0.1% is larger than that of the top 1%; and the top 1% also gains larger welfare than the top 1%-10%.
According to the definition of welfare in the equation (6.1), there are two factors that are behind the change of welfare: the first is the value \( V(\bullet) \); and the second is the distribution \( H(\bullet) \). We call the first factor the *value effect*; and the second *distribution effect*. In our model, the distribution effect dominates the value effect. For example, the top-left panel of the Figure 6.2 indicates that given \( a^i \) not large enough, for either type, the household’s value increases when \( \theta \) drops.\(^{36}\) Yet the bottom-90% wealth group see their welfare reducing. This is due to the fact that decreasing \( \theta \) shifts the mode of the wealth distribution to the left.\(^{37}\)

\(^{36}\) With all calibrations that are considered in this paper, H-type’s value is still decreasing in \( \theta \) way after \( a^i \) passes the threshold of the top-10% wealth group.

\(^{37}\) The decrease in the welfare for the bottom-90% of the household is mainly driven by the L-type households. With our calibration, when \( \theta \) decreases, welfare barely changes for the H-type households.
7 Conclusions

In this paper, we first present the empirical evidence on the decrease in the intermediation cost of borrowing. Using a two-way error component panel-data model with fixed individual and time effects, we regress the cost of borrowing on firm’s characteristics that determine the borrowing amount and the risk premium, and abstract the estimates of the time effects. The estimation results imply that the cost of borrowing has decreased substantially from the year 1980 to 2007, due to the decrease in the cost of credit supply. We further regress the estimated time effects of the credit-supply cost on the real interest rate. The residuals of the regression measure the movement in the intermediation cost. We find that, because of the decrease in the intermediation cost, the cost of borrowing in the corporate sector of the United States has dropped by 13%-17%.

We then build up a dynamic general equilibrium framework to explore the macroeconomic consequences of the decreasing intermediation cost. We find that the reduction in the intermediation cost triggers two feedback loops. The first is between the capital and credit market, and the second is between the capital-credit market and the wealth distribution. The two feedback loops amplify the initial impact of the decreasing intermediation cost. We show that after the reduction in the intermediation cost, credit market experiences a “simultaneous” expansion from both credit-demand and credit-supply side. Because of the “simultaneous” nature, the real risk-free interest rate barely changes. In the meantime, the capital market also sees an expansion. But the extent of the capital-market expansion is less significant than the credit-market expansion. The feedback loop between the capital and credit market exaggerates the capital-income risk and increases the average returns to investment among the leveraged investors (H-types), and hence the wealth heterogeneity increases. We find that the overall inequality in income and wealth rises, albeit the income inequality among L-type households drops. In terms of welfare, the decrease in the intermediation cost improves the overall welfare, as well as the welfare of the wealthy households, although the bottom-90% households in terms of wealth see their welfare decreased.

Our framework suggests that the decrease in the intermediation cost is behind the stylized facts of credit expansion, the secular rise in the income and wealth inequality, as well as the secular fall in the asset returns. For the past decades, the real interest rate in the United States also saw a decreasing trend. Although we cannot capture such trend with our current framework, the real risk-free interest rate in our model barely changes when the intermediation cost decreases. There are many factors that could affect the real interest rate, such as the global saving glut, central bank’s policies, etc. We maintain that these factors, instead of the intermediation cost, are behind the recent secular decrease in the real interest rate.
There are several aspects on which our work can still be extended. One of the interesting extensions might be an inclusion of the aggregate shock. This enables an assessment of the impact of the intermediation cost on the dynamics of the economy, especially the dynamics of the asset price. Furthermore, the transitional dynamics associated with changing intermediation cost could also be explored.
A Details on the Empirical Analysis

A.1 Independent Variables

As discussed in the main text, independent variables of the regression model (2.1) are specified to explain the firm’s revenue, amount of borrowing, as well as its risk premium. These variables can be grouped into six categories: liquidity, capital structure, solvency, profitability, market valuation and others. Table A.1 lists the definitions of the independent variables.

We expect liquidity to be negatively correlated with the amount of borrowing and the risk premium. On one hand, low liquidity is associated with high burden of (the short-term) debt. On the other hand, if the liquidity is low, lending to the firm is risky. In this case, creditors usually charge a high risk premium. Since the firm with lower liquidity has higher amount of borrowing and usually has higher risk premium, the cost of borrowing of the firm must be higher. We use three variables to measure the liquidity from different angles, and we expect negative signs in the regression results on these variables.

Three variables are specified to measure the capital structure. The proportion of debt in current liabilities matters for the near-term interest payment. Since more short-term debt implies higher interest expense in the next-period, we expect a positive sign in the regression results on the lag-1 of the variable. Moreover, as firm finances its business through three channels: short-term debt, long-term debt, and capital, we use two more variables to describe the composition of the three components of finance. We expect the effect of the debt ratio to be similar as for the liquidity measures. However, it is hard to predict the sign on the short- to long-term debt ratio, as it depends on firm’s short-term and long-term borrowing contract.

For the solvency of firm, we specify two measures involving the flows from the income statement. The effect of the two variables on the cost of borrowing is ambiguous. One one hand, if there is less borrowing and if the risk premium is low, income corresponds to less debt obligations and thus the solvency is high. But on the other hand, increasing borrowing usually implies an expansion of production, which generally results in higher income. In the latter case, more income is available to meet the debt obligations, and thus the solvency might also be high. In the end, we cannot be sure on the signs of the regression results on the two solvency measures involving flows.
<table>
<thead>
<tr>
<th>Variable names</th>
<th>Variable definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Liquidity</strong></td>
<td></td>
</tr>
<tr>
<td>Current ratio</td>
<td>= Current assets / Current liabilities.</td>
</tr>
<tr>
<td>Quick ratio</td>
<td>= (Current assets − Inventories) / Current liabilities.</td>
</tr>
<tr>
<td>Cash ratio</td>
<td>= Cash and short-term investments / Current liabilities.</td>
</tr>
<tr>
<td><strong>Capital Structure</strong></td>
<td></td>
</tr>
<tr>
<td>Prop.debt in current liabilities</td>
<td>= Debt in current liabilities / Current liabilities.</td>
</tr>
<tr>
<td>Short- to long-term debt ratio</td>
<td>= Debt in current liabilities / Total debt</td>
</tr>
<tr>
<td>Debt ratio</td>
<td>= Total debt / Total assets.</td>
</tr>
<tr>
<td><strong>Solvency</strong></td>
<td></td>
</tr>
<tr>
<td>Solvency ratio</td>
<td>= (Net income + Depreciation and Armotization) / Total liabilities.</td>
</tr>
<tr>
<td>Interest coverage ratio</td>
<td>= EBIT / Interest and related expense.</td>
</tr>
<tr>
<td><strong>Profitability</strong></td>
<td></td>
</tr>
<tr>
<td>Gross margin</td>
<td>= (Revenue − Cost of goods sold) / Revenue.</td>
</tr>
<tr>
<td>Return on assets (ROA)</td>
<td>= Net income / Total assets.</td>
</tr>
<tr>
<td><strong>Market valuation</strong></td>
<td></td>
</tr>
<tr>
<td>Price-earning ratio (PE)</td>
<td>= Stock price (close) / EPS (12-month Moving average).</td>
</tr>
<tr>
<td>Price-to-book ratio (PB)</td>
<td>= (Stock price (close) × Common shares outstanding) / Book value.</td>
</tr>
<tr>
<td><strong>Others</strong></td>
<td></td>
</tr>
<tr>
<td>Size</td>
<td>First obtain the Market value = Share price (close) × Common shares outstanding.</td>
</tr>
<tr>
<td></td>
<td>Denote this market value as $V_t$. For year $t$, denote the largest and smallest</td>
</tr>
<tr>
<td></td>
<td>market value among all firms as $\overline{V}_t$ and $\underline{V}_t$, respectively.</td>
</tr>
<tr>
<td></td>
<td>The size of the firm in year $t$ is calculated as $(V_t - \underline{V}_t)/(\overline{V}_t - \underline{V}_t)$.</td>
</tr>
<tr>
<td>Tangibility</td>
<td>Property, Plant, and Equipment (Net) / Total assets.</td>
</tr>
</tbody>
</table>
Variable names | Variable definitions
--- | ---
Change of LT debt-equity | Percentage-change of the Long-term debt-to-equity from the last year, where
Long-term debt-to-equity = Long-term debt / Equity.
Change of ST debt-equity | Percentage-change of the Short-term debt-to-equity from the last year, where
Short-term debt-to-equity = Debt in current liabilities / Equity.

Notes: Total debt = Debt in current liabilities + Long-term debt. EBIT stands for Earnings Before Interest and Taxes, which is defined as follows: EBIT = Operating income after depreciation + Non-operating income (expense); The definition of book value follows Fama and French (1993): Book value = Equity + Deferred tax and investment tax credit − (Book value of) Preferred stock.

It is also hard to predict the relationship between the profitability of a firm and its cost of borrowing. On one hand, more profitable firms have the motivation to increase borrowing; but on the other hand, they usually have lower risk premia. The impact of the firm’s market valuation on the cost of borrowing is also ambiguous, because the market valuation can either signal risks or proxy for additional values (Graham et al., 2008). Following Lin et al. (2011), we include the size and the tangibility of the firm’s assets into our regression, and we expect the two to be negatively correlated with the risk premium. However, higher tangibility makes it easier for a firm to borrow, and therefore, the amount of borrowing might be higher if a firm owns more tangible assets. Hence, the impact of the tangibility on the cost of borrowing is mixed. Finally, we include the percentage-changes of debt-to-equity ratios from the last period into our regression, in order to control for the short-term variation of debt.

A.2 Descriptive Statistics

Table A.2 reports the descriptive statistics of the dependent and independent variables. Comparing the balanced with the unbalanced panel, the latter has larger standard deviations on all variables. Moreover, for variables involving items of income-statement (the flows), the differences between the mean and median are much larger in the unbalanced panel than that in the balanced panel. This is due to the fact that the unbalanced panel has a large number of outliers. Many firms that dropped out of the sample in the middle of the time window reported very low earnings and revenues, as well as very high costs of borrowing. Some new firms, belonging to the “new economy” such as the internet sector,
Table A.2. Descriptive statistics.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Balanced Panel</th>
<th></th>
<th></th>
<th></th>
<th>Unbalanced Panel</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Median</td>
<td>Std.</td>
<td>N</td>
<td>Mean</td>
<td>Median</td>
<td>Std.</td>
<td>N</td>
</tr>
<tr>
<td>Dependent variable</td>
<td>4.8649</td>
<td>2.8132</td>
<td>6.8305</td>
<td>11078</td>
<td>6.2669</td>
<td>2.2889</td>
<td>11.2945</td>
<td>42382</td>
</tr>
<tr>
<td>Liquidity</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Current ratio</td>
<td>1.5563</td>
<td>1.3289</td>
<td>.9183</td>
<td>10385</td>
<td>2.2022</td>
<td>1.6323</td>
<td>2.6210</td>
<td>36915</td>
</tr>
<tr>
<td>Quick ratio</td>
<td>1.0214</td>
<td>.9045</td>
<td>.6284</td>
<td>10330</td>
<td>1.6225</td>
<td>1.0831</td>
<td>2.5155</td>
<td>36670</td>
</tr>
<tr>
<td>Cash ratio</td>
<td>.2631</td>
<td>.1200</td>
<td>.5206</td>
<td>10486</td>
<td>.7951</td>
<td>.2107</td>
<td>2.5008</td>
<td>37431</td>
</tr>
<tr>
<td>Capital Structure</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P.d. curr.liab</td>
<td>.2324</td>
<td>.2091</td>
<td>.1696</td>
<td>10388</td>
<td>.2246</td>
<td>.1742</td>
<td>.2208</td>
<td>37185</td>
</tr>
<tr>
<td>SL ratio</td>
<td>.1899</td>
<td>.1294</td>
<td>.1868</td>
<td>10903</td>
<td>.2728</td>
<td>.1619</td>
<td>.2851</td>
<td>40505</td>
</tr>
<tr>
<td>Debt ratio</td>
<td>.3025</td>
<td>.3074</td>
<td>.1371</td>
<td>10909</td>
<td>.2880</td>
<td>.2738</td>
<td>.1958</td>
<td>41304</td>
</tr>
<tr>
<td>Solvency</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solvency ratio</td>
<td>.1722</td>
<td>.1401</td>
<td>.1692</td>
<td>9872</td>
<td>.0673</td>
<td>.1428</td>
<td>4.7059</td>
<td>36839</td>
</tr>
<tr>
<td>Interest coverage ratio</td>
<td>8.8903</td>
<td>3.9400</td>
<td>35.8391</td>
<td>11070</td>
<td>18.7774</td>
<td>3.7023</td>
<td>430.6623</td>
<td>42337</td>
</tr>
<tr>
<td>Profitability</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gross margin</td>
<td>.2911</td>
<td>.2856</td>
<td>.1757</td>
<td>11078</td>
<td>.0352</td>
<td>.3023</td>
<td>6.1379</td>
<td>42370</td>
</tr>
<tr>
<td>ROA</td>
<td>.0484</td>
<td>.0459</td>
<td>.0609</td>
<td>11078</td>
<td>−.0015</td>
<td>.0418</td>
<td>1.3225</td>
<td>42331</td>
</tr>
<tr>
<td>Market valuation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PE</td>
<td>14.1173</td>
<td></td>
<td>9325</td>
<td>13.4641</td>
<td>36884</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PB</td>
<td>1.9105</td>
<td>1.3968</td>
<td>2.2425</td>
<td>8915</td>
<td>1 × 10^{14}</td>
<td>1.7379</td>
<td>2 × 10^{16}</td>
<td>35754</td>
</tr>
<tr>
<td>Others</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Size</td>
<td>.0383</td>
<td>.0096</td>
<td>.0980</td>
<td>9332</td>
<td>.0182</td>
<td>.0018</td>
<td>.0626</td>
<td>37003</td>
</tr>
<tr>
<td>Tangibility</td>
<td>.4779</td>
<td>.4519</td>
<td>.2608</td>
<td>10961</td>
<td>.3549</td>
<td>.2945</td>
<td>.2679</td>
<td>41001</td>
</tr>
<tr>
<td>Chg. of LT debt-equity</td>
<td>.2197</td>
<td>−.0241</td>
<td>4.0744</td>
<td>10641</td>
<td>3.0005</td>
<td>−.0383</td>
<td>106.4417</td>
<td>37349</td>
</tr>
<tr>
<td>Chg. of ST debt-equity</td>
<td>1.6454</td>
<td>−.0003</td>
<td>36.3724</td>
<td>10420</td>
<td>7.3925</td>
<td>−.0327</td>
<td>613.7207</td>
<td>36575</td>
</tr>
</tbody>
</table>

Notes: “Std.” and “N” represent standard deviation and the number of observations, respectively. On the variable names, “SL ratio” represents the short- to long-term debt ratio; “P.d. curr.liab” denotes proportion of debt in current liabilities.
reported also near-zero or even negative earnings. The existence of such firms in the unbalanced panel affects the calculation of the means and increases the standard deviations.

As an example of the existence of outliers in the unbalanced panel, one notices that the mean and standard deviation of the price-to-book ratio (PB) are over $10^{14}$. This is because there are many firms in the unbalanced panel having near-zero book values. Zero book-value is abnormal, which usually signals the coming bankruptcy. Those firms with almost-zero book values are not in the balanced panel, since the latter contains only firms without problem of going concerns. Our methodology is to manipulate the dataset as little as possible, hence we do not kick the “abnormal” firms out of the unbalanced panel. Rather, we compare the regression results of the unbalanced panel with that of the balanced panel, which do not have the issue of outliers. In Subsection 2.2 of the main text, as well as Appendix A.3 below, one will see that the existence of outliers in the unbalanced panel does not affect our empirical conclusions.

A.3 Regression Results

A.3.1 Estimation Results of the Model (2.1)

Table A.3 reports the regression results of the model (2.1). The estimation results of the fixed time effects are discussed in the main text. Here we discuss briefly the results on the controls. First of all, as expected, the coefficients for the current and quick ratios, as well as the proportion of debt in current liabilities are negative (if significant). However, unexpectedly, the cash ratio has a significant positive correlation with the cost of borrowing. A potential explanation is that although the cash ratio measures the liquidity of a firm, firms with higher interest expenses usually hold larger quantities of cash equivalents.

Regarding the capital structure, current-period proportion of debt in current liabilities is positively correlated with the cost of borrowing in the next period, due to the delayed payback of the short-term debt. Further, as expected, the results on the short- to long-term debt ratio are mixed. Finally, the explanatory power of the debt ratio to the cost of borrowing is the most significant, as can be read from the wedge between the estimated coefficients and its standard deviations. This is intuitive, because the debt ratio is a direct measure of leverage; and the leverage is immediately related to the amount of borrowing and the risk premium.

As expected, the regression on the two measures of solvency, as well as on the measures of profitability yields mixed results. The market valuation has a negative correlation with the borrowing cost, indicating that in our sample, a higher market value is associated with a lower risk premium of
Table A.3. Regression results of the model (2.1).

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Balanced</th>
<th>Unbalanced</th>
<th>Regressor</th>
<th>Balanced</th>
<th>Unbalanced</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current ratio</td>
<td>−.1643 (.2499)</td>
<td>−.2071 (.1347)</td>
<td>Current ratio</td>
<td>−.1798 (.2452)</td>
<td>−.0835 (.1398)</td>
</tr>
<tr>
<td>Quick ratio</td>
<td>−1.6151*** (.3982)</td>
<td>−.3080 (.2125)</td>
<td>Quick ratio</td>
<td>.2804 (.3911)</td>
<td>−.1879 (.2176)</td>
</tr>
<tr>
<td>Cash ratio</td>
<td>2.6283*** (.2658)</td>
<td>.6709*** (.1349)</td>
<td>Cash ratio</td>
<td>.2381 (.2680)</td>
<td>.4953*** (.1329)</td>
</tr>
<tr>
<td>P.d. curr.liab</td>
<td>−1.0984*** (.4361)</td>
<td>−.6147* (.3573)</td>
<td>P.d. curr.liab</td>
<td>.7402* (.4420)</td>
<td>.6284* (.3670)</td>
</tr>
<tr>
<td>SL ratio</td>
<td>−.7713* (.4104)</td>
<td>.4813* (.2512)</td>
<td>SL ratio</td>
<td>−.3639 (.4234)</td>
<td>−.9422*** (.2742)</td>
</tr>
<tr>
<td>Debt ratio</td>
<td>13.1322*** (.7118)</td>
<td>12.6771*** (.4659)</td>
<td>Debt ratio</td>
<td>2.5135*** (.7034)</td>
<td>1.4336*** (.4674)</td>
</tr>
<tr>
<td>Solvency ratio</td>
<td>1.5015*** (.6393)</td>
<td>.6744*** (.0833)</td>
<td>Solvency ratio</td>
<td>−2.3031*** (.6492)</td>
<td>.2406*** (.0875)</td>
</tr>
<tr>
<td>Interest coverage</td>
<td>.0006 (.0009)</td>
<td>.0003*** (.0001)</td>
<td>Interest coverage</td>
<td>.0015 (.0015)</td>
<td>.0004 (.0002)</td>
</tr>
<tr>
<td>Gross margin</td>
<td>.0420 (.3242)</td>
<td>−.3112*** (.0125)</td>
<td>Gross margin</td>
<td>.4979 (.3135)</td>
<td>−.1191*** (.0150)</td>
</tr>
<tr>
<td>ROA</td>
<td>−5.0522*** (1.3161)</td>
<td>−1.7348*** (.1628)</td>
<td>ROA</td>
<td>10.5291*** (1.3084)</td>
<td>−.7455*** (.1692)</td>
</tr>
<tr>
<td>PE</td>
<td>−2.32E−19 (.186E−19)</td>
<td>−8.16E−19*** (2.89E−19)</td>
<td>PE</td>
<td>−1.78E−19 (.186E−19)</td>
<td>−5.73E−19*** (2.87E−19)</td>
</tr>
<tr>
<td>PB</td>
<td>−.0673*** (.0173)</td>
<td>−.0043*** (.0019)</td>
<td>PB</td>
<td>−.0554*** (.0176)</td>
<td>5.28E−05 (2.12E−04)</td>
</tr>
<tr>
<td>Size</td>
<td>.0417 (1.3831)</td>
<td>.2408 (1.8035)</td>
<td>Size</td>
<td>.8579 (1.3048)</td>
<td>.7756 (1.6865)</td>
</tr>
<tr>
<td>Tangibility</td>
<td>2.7648*** (.8203)</td>
<td>1.2330** (.5918)</td>
<td>Tangibility</td>
<td>−.8303 (.8096)</td>
<td>.2952 (.5818)</td>
</tr>
</tbody>
</table>
borrowing. Finally, the tangibility positively affects the borrowing cost, because the firm with more tangible assets is able to borrow more from the credit market.

**A.3.2 Estimation Results of the Model (2.2)**

The real risk-free rate, as the independent variable of the benchmark regression of the model (2.2), is the 1-Year treasury constant maturity rate deflated by the CPI. Table A.4 reports the benchmark estimation results. Not surprisingly, the real risk-free rate has a significant explanatory power to the time effects of the credit-supply cost. The adjusted $R^2$ is not high, indicating that a large proportion of the variation in the dependent variable is not explained by the regressor. In our framework, this part of the unexplained variation is due to the change in the intermediation cost, which is captured by the residuals. The estimation results on the residuals are discussed already in the main text.

As a robustness check, we use different measures of the risk-free rate and inflation as dependent variables of the model (2.1). To be specifically, we proxy the risk-free rate also by the effective federal fund rate and the US dollar-denominated London Interbank Offered Rate (LIBOR). In addition, we measure inflation also by the personal consumption expenditure (PCE) price index as well as the GDP
**Table A.4.** Regression results of the model (2.2).

<table>
<thead>
<tr>
<th>Regressor</th>
<th>Balanced</th>
<th>Unbalanced</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Risk-free Rate</td>
<td>.1787***</td>
<td>.1505***</td>
</tr>
<tr>
<td></td>
<td>(.0418)</td>
<td>(.04523)</td>
</tr>
<tr>
<td>(Intercept)</td>
<td>−.4065***</td>
<td>−.3424**</td>
</tr>
<tr>
<td></td>
<td>(.1256)</td>
<td>(.1359)</td>
</tr>
<tr>
<td>Nr. Period</td>
<td>26</td>
<td>26</td>
</tr>
<tr>
<td>Adj. R²</td>
<td>.4087</td>
<td>.2872</td>
</tr>
</tbody>
</table>

*Notes:* Numbers in parentheses denote standard deviation of the corresponding estimates. Significance at the 10%, 5%, and 1% level is indicated by *, ** and ***, respectively.

deflator. Due to the mismatch of durations,\(^{38}\) when we use the federal fund rate and the LIBOR rate as proxies for the risk-free rate, we cannot simply subtract the interest rate by the percentage change of the price index to obtain the real interest rate. In such case, we put both the (nominal) risk-free rate and the price index as independent variables. Table A.5 and A.6 report the results of such robustness check for the balanced and unbalanced panel, respectively. As we can see from the tables, for all possible combinations of the risk-free rate and inflation, the linear time effect is significantly negative. This implies that the residuals of the regression (2.2) follow a significantly negative trend in all specifications. The results confirm that the effect of the decreasing intermediation cost on the cost of borrowing is robust against different measures of risk-free rate and inflation.

To verify the validity of our results and methodology, we use the Moody’s Aaa-graded corporate bond yield as the regressor and re-run the regression of the model (2.2). The yield to the Aaa-graded corporate bond carries little default-risk premium. But unlike the risk-free rates that we have used so far, such yield contains the intermediation cost. If our methodology is right, there should not exist a clear trend of the residuals. This is because all the variation in the dependent variable should be fully

---

\(^{38}\) The effective federal fund rate is the interest rate for the overnight financing, hence there is no price index that can match such a short duration. For LIBOR, we use the 3-month USD LIBOR rate. To rule out the seasonality of the price index, we also do not deflate the LIBOR rate by quarterly inflation. The estimation results (Table A.5 and A.6) show that the inflation rate does not have explanatory power to the dependent variable, and it does not matter to our empirical results whether we deflate the interest rate by inflation or put the price index directly as an additional independent variable.
Table A.5. Results with different measures of risk-free rate and inflation for model (2.2), Balanced Panel.

<table>
<thead>
<tr>
<th>Regressor</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>bond yield - PCE</td>
<td>.1753***</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(.0446)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>bond yield - Def</td>
<td></td>
<td>.1827***</td>
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<td></td>
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<td>(.0404)</td>
<td></td>
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</tr>
<tr>
<td>Fed fund rate</td>
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<td>.1659***</td>
<td>.1494***</td>
<td>.1583***</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>(.0370)</td>
<td>(.0374)</td>
<td>(.0365)</td>
<td></td>
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</tr>
<tr>
<td>LIBOR</td>
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<td></td>
<td>.1308**</td>
<td>.1233**</td>
<td>.1311***</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>(.0471)</td>
<td>(.0464)</td>
<td>(.0446)</td>
</tr>
<tr>
<td>CPI</td>
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<td></td>
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<td>(.1062)</td>
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</tr>
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<td>PCE</td>
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<tr>
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<td>26</td>
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<td>Adj. $R^2$</td>
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<td>Linear time effect</td>
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</tr>
</tbody>
</table>

Notes: “LIBOR” stands for dollar-denominated 3-month LIBOR rate. “Def” represents the GDP-deflator. Numbers in parentheses denote standard deviation of the corresponding estimates. Significance at the 10%, 5%, and 1% level is indicated by *, ** and ***, respectively.
Table A.6. Results with different measures of risk-free rate and inflation for model (2.2), Unbalanced Panel.

<table>
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<tr>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>bond yield - PCE</td>
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<td></td>
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<tr>
<td>bond yield - Def</td>
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<td>.1543***</td>
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<td></td>
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<td>Fed fund rate</td>
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<td>-.9803***</td>
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<td>-.7664**</td>
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<td>(.1605)</td>
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<td>(.2395)</td>
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<td>(.2928)</td>
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<td>(.2999)</td>
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<td>26</td>
<td>26</td>
<td>22</td>
<td>22</td>
<td>22</td>
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<td>.3098</td>
<td>.4940</td>
<td>.5114</td>
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<td>.1858</td>
<td>.2001</td>
<td>.1838</td>
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<td>-.0205**</td>
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<td>-.0307**</td>
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<tr>
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<td>(.0097)</td>
<td>(.0097)</td>
<td>(.0087)</td>
<td>(.0086)</td>
<td>(.0086)</td>
<td>(.0111)</td>
<td>(.0109)</td>
<td>(.0110)</td>
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</table>

Notes: “LIBOR” stands for dollar-denominated 3-month LIBOR rate. “Def” represents the GDP-deflator. Numbers in parentheses denote standard deviation of the corresponding estimates. Significance at the 10%, 5%, and 1% level is indicated by *, ** and ***, respectively.
covered by the change in the corporate bond yield. The results of the regression, as reported by Table A.7, confirm that this is indeed the case. The linear time effect is almost zero and non-significant, implying that there is no time trend on the residuals.

Table A.7. Regression results with the Aaa corporate bond rate for the model (2.2).

<table>
<thead>
<tr>
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<th></th>
<th>Unbalanced</th>
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</tr>
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</tr>
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<td>Aaa rate</td>
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<td>.2221***</td>
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<td>(.0315)</td>
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<td>(.0335)</td>
</tr>
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<td>CPI</td>
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<td>−.0163</td>
</tr>
<tr>
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<td>(.0607)</td>
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<td></td>
<td>(.0645)</td>
</tr>
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<td>.0193</td>
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<td>(.0634)</td>
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<td></td>
<td>(.0669)</td>
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<td>Def</td>
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<tr>
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<td>(.0731)</td>
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<td></td>
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</tr>
<tr>
<td>(Intercept)</td>
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<td>−1.7748***</td>
<td>−1.7114***</td>
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<td>(.2142)</td>
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<td>(.2278)</td>
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<td>26</td>
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<td>Adj. R²</td>
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</table>

Notes: “Aaa rate” stands for Aaa-graded corporate bond yield. “Def” represents the GDP-deflator. Numbers in parentheses denote standard deviation of the corresponding estimates. Significance at the 10%, 5%, and 1% level is indicated by *, ** and ***, respectively.
B Proofs for the Two-Period Model

B.1 Proof to the Lemma 3.1

We first show that the demand-side of the credit market exists. The bank is willing to lend to the H-type and charge the interest rate $\pi$ based on (3.5), if the H-type can pay back the principal plus interest rate when capital formation is successful (when $\varepsilon = 1$). Namely, if $R^c i_H \geq (1 + \pi)b$. By equation (3.5), this implies a bank’s lending condition of

$$
(1 - \lambda)R^c \geq (1 + r^f)(1 + \theta).
$$

(B.1)

This condition is weakly satisfied under (3.8). We therefore conclude that the demand-side of the credit market exits in equilibrium.

Next, we look at the supply side of the credit market. To start, it is obvious that borrowing to invest is not optimal for the L-type. Since the model doesn’t allow the agent to borrow and save simultaneously, L-type only borrows when she invests more than $w$. But when she does this, she would receive zero-consumption when the capital formation fails. With the log utility, zero-consumption results to an utility of minus infinity. Therefore, any $i_L > w$ is suboptimal. When the L-type do not borrow to invest, her optimal amount of investment is given by equation (3.10). If $0 < i_L < w$, it must be satisfied that

$$
0 < \frac{\lambda R^c}{R^c - (1 + r^f)} < 1,
$$

which translate to the condition of

$$
(1 - \lambda)R^c > 1 + r^f,
$$

(B.2)

which always holds given the $R^c$ in (3.8). We can thus conclude that in equilibrium, L-type always allocates some of her endowment to saving. Hence the supply side of the credit thus also exists.  

B.2 Proof to the Proposition 3.2

In general equilibrium, (3.8) holds; and the equation (3.17) of the proposition is just a re-arrangement of (3.8). We take the $R^c$ implied by (3.8) to the L-type’s first-order condition (3.10) and obtain

$$
i_L = \left[1 - \frac{\lambda(1 + \pi)(1 + \theta)}{(1 + \pi)(1 + \theta) - (1 + r^f)}\right] w,

$$

(B.3)

which, after taking in the interest-rate scheme (3.5) and some simple re-arrangement, leads to the equation (3.11) of the proposition. Because $\theta(1 - \lambda) > 0$ by the construction of the model,

$$
0 < \frac{\lambda(1 + \theta)}{\theta + \lambda} < 1,
$$

60
and hence \(0 < i_L < w\). Therefore the amount of saving by the L-type is \(s = w - i_L\), which is given by the equation (3.14) of the proposition. The aggregate saving of the economy (from the L-type) is thus

\[
S = (1 - p)s = (1 - p) \frac{\lambda(1 + \theta)}{\theta + \lambda} w,
\]

which is exactly the equation (3.16) of the proposition.

The aggregate borrowing of the H-type is \(pb = p(1 + \theta)i_H\). In general equilibrium, credit market clears. The aggregate borrowing of the H-type must equal to \(S\), namely.

\[
p(1 + \theta)i_H = S = (1 - p) \frac{\lambda(1 + \theta)}{\theta + \lambda} w,
\]

from which we obtain

\[
i_H = \frac{1 - p}{p} \cdot \frac{\lambda}{\theta + \lambda} w,
\]

which is the equation (3.12) of the proposition; \(b = (1 + \theta)i_H\) is thus obtained as the equation (3.13) of the proposition. Moreover, from the equation (3.2) we obtain another representation of \(i_H\):

\[
i_H = \frac{K - (1 - p)(1 - \lambda)i_L}{p(1 - \lambda)}.
\]

Equating the two representations of the \(i_H\) in equations (B.5) and (B.6),

\[
\frac{1 - p}{p} \cdot \frac{\lambda}{\theta + \lambda} w = \frac{K - (1 - p)(1 - \lambda)i_L}{p(1 - \lambda)},
\]

and taking the representation of \(i_L\) in equation (3.11) to (B.7) above and solving for \(K\), the equation (3.15) of the proposition is obtained.

\[\blacksquare\]

**B.3 Proof to the Proposition 3.3**

Based on the equation (3.17) of the Proposition 3.2, in general equilibrium, we have

\[
1 + r^f = \frac{1 - \lambda}{1 + \theta} K^{\alpha - 1}.
\]

The first-order partial derivative of \(r^f\) with respect to \(\theta\) is

\[
\frac{\partial r^f}{\partial \theta} = (1 - \lambda) \frac{(\alpha - 1)(1 + \theta)K^{\alpha - 2}\frac{\partial K}{\partial \theta} - K^{\alpha - 1}}{(1 + \theta)^2}
\]

\[
= \frac{(1 - \lambda)K^{\alpha - 2}}{(1 + \theta)^2} \left[(\alpha - 1)(1 + \theta)\frac{\partial K}{\partial \theta} - K\right].
\]
Hence \( \frac{\partial f}{\partial \theta} < 0 \) if \( u := (\alpha - 1)(1 + \theta)\frac{\partial K}{\partial \theta} - K < 0 \). From the equation (3.15) we see that

\[
\frac{\partial K}{\partial \theta} = -(1 - p)(1 - \lambda)w \cdot \frac{\lambda^2}{(\theta + \lambda)^2}, \tag{B.9}
\]

and hence

\[
u = (1 - \alpha)(1 + \theta)(1 - p)(1 - \lambda)w \cdot \frac{\lambda^2}{(\theta + \lambda)^2} - (1 - p)(1 - \lambda) \left[ 1 - \frac{\lambda \theta}{\theta + \lambda} \right] w
\]

\[
= (1 - p)(1 - \lambda)w \left[ (1 - \alpha)(1 + \theta) \cdot \frac{\lambda^2}{(\theta + \lambda)^2} + \frac{\lambda \theta}{\theta + \lambda} - 1 \right] \tag{B.10}
\]

\[
= (1 - p)(1 - \lambda)w \cdot \frac{-\alpha(1 + \theta)\lambda^2 - \theta(1 - \lambda)(\theta + 2\lambda)}{(\theta + \lambda)^2} < 0.
\]

Therefore, the \( r^f \) is an decreasing function of \( \theta \). The Proposition 3.3 follows. ■

### B.4 Proof to the Proposition 3.4

Denote by \( L \) the L-type’s objective function, and take the \( c^1_L \) and \( c^0_L \) (eq. (3.11), (3.14)) in the general equilibrium to \( L \):

\[
L = (1 - \lambda) \log c^1_L + \lambda \log c^0_L = (1 - \lambda) \log \left( (1 - \lambda)R^c w \right) + \lambda \log \left( \frac{\lambda}{\theta + \lambda} (1 - \lambda)R^c w \right).
\]

The first-order derivative of \( L \) with respect to \( \theta \) writes

\[
\frac{\partial L}{\partial \theta} = (1 - \lambda) \frac{(1 - \lambda)w \cdot \frac{\partial R^c}{\partial \theta} + \lambda(1 - \lambda)w \cdot \frac{\frac{\partial R^c(\theta + \lambda)}{\partial \theta} - R^c}{(\theta + \lambda)^2} }{(1 - \lambda)R^c w} + \lambda \frac{\frac{\lambda}{\theta + \lambda}(1 - \lambda)R^c w}{(1 - \lambda)R^c w}
\]

\[
= (1 - \lambda) \cdot \frac{1}{R^c} \cdot \frac{\partial R^c}{\partial \theta} + \lambda \cdot \frac{1}{R^c} \cdot \left[ \frac{\partial R^c(\theta + \lambda)}{\partial \theta} - R^c \right]
\]

\[
= \frac{1}{R^c} \cdot \frac{\partial R^c}{\partial \theta} - \frac{\lambda}{\theta + \lambda}
\]

\[
= \frac{1}{K^{\alpha - 1}} \cdot (\alpha - 1)K^{\alpha - 2} \cdot \frac{\partial K}{\partial \theta} - \frac{\lambda}{\theta + \lambda} \quad \text{(since } R^c = K^{\alpha - 1} \text{ in equilibrium)}
\]

\[
= (\alpha - 1) \frac{1}{K} \cdot \frac{\partial K}{\partial \theta} - \frac{\lambda}{\theta + \lambda}
\]

\[
= (1 - \alpha) \frac{1}{(1 - p)(1 - \lambda)} \left[ 1 - \frac{\lambda \theta}{\theta + \lambda} \right] w \cdot (1 - p)(1 - \lambda)w \cdot \frac{\lambda^2}{(\theta + \lambda)^2} - \frac{\lambda}{\theta + \lambda} \quad \text{(eq. (3.15) and (B.9))}
\]

\[
= \left[ \frac{(1 - \alpha)\lambda}{(1 - \theta)\lambda + \theta - 1} \right] \cdot \frac{\lambda}{\theta + \lambda} < 0.
\]
The last line follows from the fact that

\[ 0 > -\alpha \lambda - (1 - \lambda)\theta = (1 - \alpha)\lambda - [(1 - \theta)\lambda + \theta]; \]

\[ \iff \frac{(1 - \alpha)\lambda}{(1 - \theta)\lambda + \theta} < 1. \]

Hence we show that the welfare of the L-type is decreasing in \( \theta \). Decreasing \( \theta \) would thus improve the welfare of the L-type. ■
C Proofs for the Dynamic Model

For the succinctness of presentation, in all proofs of this section, I drop the individual subscript $i$.

C.1 Proof to the Proposition 4.1

The borrower repays all the debt plus interest if

$$(1 - \varphi)(1 + r_{t+1}^c)\exp\{\eta_t + \varepsilon_t\}i_t^\nu \geq -(1 + \pi_t)s_t;$$

$$\iff \varepsilon_t \geq \log\left(\frac{-\frac{(1 + \pi_t)s_t}{(1 - \varphi)(1 + r_{t+1}^c)i_t^\nu \exp\{\eta_t\}}}{0}\right);$$

namely $\varepsilon_t \geq \log\left((1 + \pi_t)L_t\right)$.

For the succinctness of notation, in the following, we define $E(\pi_t) := \log\left((1 + \pi_t)L_t\right)$. Given $\pi_t > 0$, we can thus write the expected reclaim from the bank as

$$\int_{-\infty}^{\infty} b_{t+1} f(\varepsilon_t) d\varepsilon_t = -\int_{-\infty}^{\infty} (1 + \pi_t)s_t f(\varepsilon_t) d\varepsilon_t + \int_{-\infty}^{\infty} (1 - \varphi)(1 + r_{t+1}^c)\exp\{\eta_t + \varepsilon_t\}i_t^\nu f(\varepsilon_t) d\varepsilon_t \quad \text{(C.2)}$$

$$= -(1 + \pi_t)s_t \int_{-\infty}^{\infty} f(\varepsilon_t) d\varepsilon_t + (1 - \varphi)(1 + r_{t+1}^c)i_t^\nu \exp\{\eta_t\} \int_{-\infty}^{\infty} \exp\{\varepsilon_t\} f(\varepsilon_t) d\varepsilon_t.$$

Remember that $\varepsilon_t \sim N(-\sigma^2/2, \sigma^2)$, hence

$$\int_{-\infty}^{\infty} f(\varepsilon_t) d\varepsilon_t = 1 - \Phi\left(\frac{E(\pi_t) + \frac{\sigma^2}{2}}{\sigma}\right). \quad \text{(C.3)}$$

Further,

$$\int_{-\infty}^{\infty} \exp\{\varepsilon_t\} f(\varepsilon_t) d\varepsilon_t = \int_{-\infty}^{\infty} \exp\{\varepsilon_t\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\varepsilon_t + \frac{\sigma^2}{2})^2}{2\sigma^2}\right\} d\varepsilon_t$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\varepsilon_t - \frac{\sigma^2}{2})^2}{2\sigma^2}\right\} d\varepsilon_t \quad \text{(C.4)}$$

$$= \Phi\left(\frac{E(\pi_t) - \frac{\sigma^2}{2}}{\sigma}\right).$$

Substituting the corresponding terms in the equation (C.2) by the equation (C.3) and (C.4) and we obtain

$$\int_{-\infty}^{\infty} b_{t+1} f(\varepsilon_t) d\varepsilon_t = -(1 + \pi_t)s_t \left[1 - \Phi\left(\frac{E(\pi_t) + \frac{\sigma^2}{2}}{\sigma}\right)\right] + (1 - \varphi)(1 + r_{t+1}^c)i_t^\nu \exp\{\eta_t\} \Phi\left(\frac{E(\pi_t) - \frac{\sigma^2}{2}}{\sigma}\right).$$
Take the equation above back to the equation (4.17) and rearrange,

\[
1 + r_t^f = (1 + \pi_t) \left[ 1 - \Phi \left( \frac{\mathcal{E}(\pi_t) + \sigma^2/2}{\sigma} \right) \right] - \frac{(1 - \varphi)(1 + r_{t+1}^c)i_t'}{s_t} \exp \{ \eta_t \} \Phi \left( \frac{\mathcal{E}(\pi_t) - \sigma^2/2}{\sigma} \right). \tag{C.5}
\]

Using the definition of \( \mathcal{E}(\pi_t) \) and the \( L_t \) to group the corresponding terms in the equation (C.5) and the equation (4.18) is obtained. \( \blacksquare \)

### C.2 Proof to the Proposition 4.2

Below, we define the right-hand side of the equation (4.18) as a function of \( \pi_t \) and \( L_t \), and denote this function as \( B(\pi_t, L_t) \). Namely,

\[
B(\pi_t, L_t) = (1 + \pi_t) \left[ 1 - \Phi \left( \frac{\log ((1 + \pi_t)L_t) + \sigma^2/2}{\sigma} \right) \right] + L_t^{-1} \Phi \left( \frac{\log ((1 + \pi_t)L_t) - \sigma^2/2}{\sigma} \right). \tag{C.6}
\]

The next lemma shows that for any \( L_t > 0 \), \( B(\cdot) \) is a monotonically increasing function in \( \pi_t \in (0, \infty) \).

**Lemma C.1** Given \( L_t > 0 \), the function \( B(\cdot) \) is a monotonically increasing function in \( \pi_t \in (0, \infty) \). Moreover,

\[
\frac{\partial B(\cdot)}{\partial \pi_t} = 1 - \Phi \left( \frac{\log ((1 + \pi_t)L_t) + \sigma^2/2}{\sigma} \right) > 0. \tag{C.7}
\]

**Proof** For any \( L_t > 0 \) we have \( \pi_t > 0 \), the first-order derivative can be calculated as follows:

\[
\frac{\partial B(\cdot)}{\partial \pi_t} = \left[ 1 - \Phi \left( \frac{\log ((1 + \pi_t)L_t) + \sigma^2/2}{\sigma} \right) \right] - \frac{1}{\sigma} \cdot \phi \left( \frac{\log ((1 + \pi_t)L_t) + \sigma^2/2}{\sigma} \right) + \frac{L_t^{-1}}{\sigma(1 + \pi_t)} \cdot \phi \left( \frac{\log ((1 + \pi_t)L_t) - \sigma^2/2}{\sigma} \right) =: C, \tag{C.8}
\]

where \( \phi(\cdot) \) is the probability density function (PDF) of the standard normal distribution. Notice that

\[
\frac{\phi \left( \frac{\log ((1 + \pi_t)L_t) + \sigma^2/2}{\sigma} \right)}{\phi \left( \frac{\log ((1 + \pi_t)L_t) - \sigma^2/2}{\sigma} \right)} = \exp \left\{ - \frac{\left( \log ((1 + \pi_t)L_t) + \sigma^2/2 \right)^2}{2\sigma^2} \right\} = \frac{1}{\exp \left\{ \log ((1 + \pi_t)L_t) \right\}} = \frac{L_t^{-1}}{(1 + \pi_t)}, \tag{C.9}
\]

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therefore, the term \( C \) in the equation (C.8) is 0. The equation (C.8) thus becomes

\[
\frac{\partial B(\bullet)}{\partial \pi_t} = 1 - \Phi \left( \frac{\log \left( (1 + \pi_t) L_t \right) + \sigma^2/2}{\sigma} \right) > 0.
\] (C.10)

Thus for any \( L_t > 0 \), the function \( B(\bullet) \) is monotonically increasing in \( \pi_t \in (0, \infty) \). ■

The range of \( \pi_t \) that we are interested in is \( [r_t^f, \infty) \). It worth looking at the value of \( B(\pi_t, L_t) \) when \( \pi_t \) approaches to the upper and lower limit of the range. Next lemma deals with this, and the conclusion of which helps to find the condition under which the root of the equation (4.18) exists.

**Lemma C.2** Given \( L_t > 0 \), we have \( B(r_t^f, L_t) < 1 + r_t^f \), and

\[
\lim_{\pi_t \to \infty} B(\pi_t, L_t) = L_t^{-1}.
\] (C.11)

**Proof** Compare the equation (4.17) and (4.18) and we can write the \( B(\pi_t, L_t) \) as

\[
B(\pi_t, L_t) = -\frac{1}{s_t} \int_{-\infty}^{\infty} b_{t+1} f(\varepsilon_t) d\varepsilon_t
\]

\[
= -\frac{1}{s_t} \int_{-\infty}^{\infty} \min \left\{ - (1 + \pi_t) s_t, (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} \right\} f(\varepsilon_t) d\varepsilon_t.
\] (C.12)

To prove the conclusion that \( B(r_t^f, L_t) < 1 + r_t^f \), we use the equation (C.2), and

\[
B(\pi_t, L_t) = \frac{1}{s_t} \left[ \int_{E(\pi_t)}^{\infty} (1 + \pi_t) s_t f(\varepsilon_t) d\varepsilon_t - \int_{-\infty}^{E(\pi_t)} (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^\nu f(\varepsilon_t) d\varepsilon_t \right].
\] (C.13)

Observe that by the condition (C.1), when \( \varepsilon_t < E(\pi_t) \), \( (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^\nu < -(1 + \pi_t) s_t \). We thus proceed from the equation (C.13) that

\[
B(\pi_t, L_t) < \frac{1}{s_t} \left[ \int_{E(\pi_t)}^{\infty} (1 + \pi_t) s_t f(\varepsilon_t) d\varepsilon_t + \int_{-\infty}^{E(\pi_t)} (1 + \pi_t) s_t f(\varepsilon_t) d\varepsilon_t \right] = (1 + \pi_t).
\]

and therefore when \( \pi_t = r_t^f \), we have \( B(r_t^f, L_t) < 1 + r_t^f \). To prove the equation (C.11), we see from the equation (C.12) that

\[
\lim_{\pi_t \to \infty} B(\pi_t, L_t) = -\frac{1}{s_t} \int_{-\infty}^{\infty} (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^\nu f(\varepsilon_t) d\varepsilon_t
\]

\[
= L_t^{-1} \int_{-\infty}^{\infty} \exp\{\varepsilon_t\} f(\varepsilon_t) d\varepsilon_t.
\]

\[
= L_t^{-1}.
\]
The last equality follows from the expectation of the lognormal distribution of \( \exp\{\varepsilon_t\} \).

With the Lemma C.1 and the Lemma C.2, we are ready to prove the Proposition 4.2.

**Proof to the Proposition 4.2** Using the notations defined above, the non-linear equation (4.17) can be written as

\[
1 + r_t^f = B(\pi_t, L_t).
\]

(C.14)

For any \( L_t > 0 \), \( B(\pi_t, L_t) \) is increasing in \( \pi_t \) (Lemma C.1). Therefore, for a given \( L_t > 0 \), there can be no more than one root of \( \pi_t \). Further, by Lemma C.2, if \( L_t < (1 + r_t^f)^{-1} \),

\[
\lim_{\pi_t \to \infty} B(\pi_t, L_t) = L_t^{-1} > (1 + r_t^f).
\]

By the same lemma, \( B(r_t^f, L_t) < 1 + r_t^f \). We can therefore conclude that root exists for \( \pi_t \in (r_t^f, \infty) \), when \( L_t \in (0, (1 + r_t^f)^{-1}) \). This proves that the function \( \Pi(L_t) \) is well-defined on \( L_t \).

When \( L_t \in L_t \), the function \( \pi_t = \Pi(L_t) \) is defined through the equation (C.14). The total differential on the equation (C.14) thus follows

\[
0 = \frac{\partial B(\bullet)}{\partial \pi_t} d\pi_t + \frac{\partial B(\bullet)}{\partial L_t} dL_t.
\]

(C.15)

The \( \frac{\partial B(\bullet)}{\partial \pi_t} \) is given by the Lemma C.1. The \( \frac{\partial B(\bullet)}{\partial L_t} \) can be obtained in a similar way as the former derivative, in which we utilize again the equation (C.9). After calculation, we have

\[
\frac{\partial B(\bullet)}{\partial L_t} = -L_t^{-2} \Phi \left( \frac{\log \left( (1 + \pi_t) L_t \right) - \sigma^2/2}{\sigma} \right).
\]

Take the formulation of \( \frac{\partial B(\bullet)}{\partial \pi_t} \) and \( \frac{\partial B(\bullet)}{\partial L_t} \) to the equation (C.15) and we can solve for the equation (4.20). The \( \frac{d\pi_t}{dL_t} \) must be greater than zero since the CDF’s must be within (0, 1), and \( L_t > 0 \) by setup.
D Technical Appendix of the Dynamic Model

This section has four subsections. In the first subsection, we characterize the relationship between consumption, investment and saving/borrowing. Especially, we describe the choice space of the household’s optimization problem, and relate the leverage and the borrowing interest rate to the choice variables. In Subsection D.2, we show that the next-period state is a well-defined continuously-differentiable function of the current-period choice variables. This property is crucial for our method to solve for the household’s optimization problem. In Subsection D.3, we derive the first-order conditions, as well as the envelope condition of the household’s dynamic optimization problem. The numerical algorithm is laid out in Subsection D.4, which is built on the conditions solved in the last subsection.

D.1 Consumption, Investment and Saving

The purpose of the discussion in this section is to reduce the number of choice variables. We show that the household’s choice of savings (borrowings), borrowing interest rate, overall investment return, as well as the gross return from saving and borrowing can be represented as functions of investment and consumption. The properties of such functions are also discussed.

D.1.1 The relationship between the investment and the leverage

In this section, we discuss the relationship between the investment and the leverage. We start from the equation (4.16). Given $a_t$ and $c_t$, such that $a_t > 0$, $c_t > 0$, and $a_t > c_t$, the equation (4.16) can be manipulated into

\[a_t - c_t - i_t = (1 - \theta L_t^\zeta)s_t;\]

\[\Leftrightarrow \frac{i_t - (a_t - c_t)}{(1 - \varphi)\left(1 + r_{t+1}^c\right)\exp\{\eta_t\}i_t^{\nu}} = (1 - \theta L_t^\zeta)L_t;\]

\[\Leftrightarrow \frac{1}{(1 - \varphi)\left(1 + r_{t+1}^c\right)\exp\{\eta_t\}}\left[i_t^{1-\nu} - (a_t - c_t)i_t^{-\nu}\right] = (1 - \theta L_t^\zeta)L_t. \tag{D.1}\]

The equation (D.1) defines a correspondence between investment $i_t$ and the leverage $L_t$, which we denote as $L_t = \mathcal{F}(i_t)$. In the next proposition, we confirm that given $a_t$ and $c_t$, $L_t = \mathcal{F}(i_t)$ is a well-defined monotonically increasing function of $i_t$ on some interval of $i_t$. The first-order derivative of the function $L_t = \mathcal{F}(i_t)$ can also be characterized analytically.

**Proposition D.1** Given $a_t$ and $c_t$ such that $a_t > 0$, $c_t > 0$ and $a_t \geq c_t$, there exist an upper bound of investment $\tilde{i}_t \in (a_t - c_t, \infty)$, such that $L_t = \mathcal{F}(i_t)$ is a one-to-one mapping from $\mathcal{I} = (a_t - c_t, \tilde{i}_t)$.
to $\mathcal{L} = (0, (1 + r_i^t)^{-1})$. The function $L_t = \mathcal{F}(i_t)$ is twice continuously differentiable and strictly increasing. Further, the derivative of the function $L_t = \mathcal{F}(i_t)$ is given by

$$\frac{dL_t}{di_t} = \frac{1}{(1 - \varphi)(1 + r_{t+1}^i)e^{\eta_i}i_t'} - \frac{(1 - \nu) + \nu(a_t - c_t)i_t^{-1}}{1 - \theta L_t^\nu(1 + \zeta)} > 0.$$  \hspace{1cm} (D.2)

**Proof** Denote the left-hand side of the equation (D.1) as $f(i_t)$, and the right-hand side of the same equation as $g(L_t)$. The equation (D.1) can thus be abbreviated by $f(i_t) = g(L_t)$. We first look at the properties of the function $f(i_t)$. The first-order derivative of $f(i_t)$ with respect to $i_t$ is

$$f'(i_t) = \frac{1}{(1 - \varphi)(1 + r_{t+1}^i)e^{\eta_i}i_t'} [(1 - \nu)i_t^{-\nu} + \nu(a_t - c_t)i_t^{-(\nu+1)}] > 0,$$

hence $f(i_t)$ is increasing in $i_t$. This implies that given $L_t$, if there is root of $i_t$ in the equation (D.1), there is at most one root. We further notice that

$$\lim_{i_t \to (a_t - c_t)^+} f(i_t) = 0 \quad \text{and} \quad \lim_{i_t \to \infty} f(i_t) = +\infty.$$  \hspace{1cm} (D.3)

Hence $f(i_t)$ is a monotonically increasing function over $(a_t - c_t, \infty)$, with the range of $(0, +\infty)$. Second, we look at the properties of the function $g(L_t)$. Again we consider the first-order derivative of the $g(L_t)$ with respect to $L_t$:

$$g'(L_t) = 1 - \theta L_t^\nu(1 + \zeta).$$

Since $\zeta$ is very small, $g'(L_t)$ is always larger than zero. This implies that given $i_t$, if there is root of $L_t$ in the equation (D.1), there is at most one root of $L_t$ in $\mathcal{L}$. We further notice that $g(0) = 0$, hence the range of $g(L_t)$ is $g(L_t) \in (0, g(\bar{L}_t))$, where $\bar{L}_t := (1 + r_i^t)^{-1}$.

Define $\tilde{i}_t$ such that $f(\tilde{i}_t) = g(\bar{L}_t)$, namely, $\tilde{i}_t$ solves

$$\frac{\tilde{i}_t - (a_t - c_t)}{(1 - \varphi)(1 + r_{t+1}^i)e^{\eta_i}i_t'} = (1 - \theta \bar{L}_t^\nu)\bar{L}_t.$$  \hspace{1cm} (D.4)

When $i_t \in \mathcal{I} = (a_t - c_t, \tilde{i}_t)$, there is always $L_t \in \mathcal{L} = (0, \bar{L}_t)$ that solves $g(L_t) = f(i_t)$, and *vice versa*. Thus the function $L_t = \mathcal{F}(i_t)$ is a well-defined one-to-one mapping from $\mathcal{I}$ to $\mathcal{L}$.

The final part of the proof concerns the first-order derivative of $L_t$ with respect to $i_t$. This is done by re-arranging the equation (D.1) into $f(i_t) - g(L_t) = 0$ and apply the total differential

$$0 = f'(i_t) di_t - g'(L_t) dL_t.$$

Taking the $f'(i_t)$ and $g'(L_t)$ derived above to the equation, the result of (D.2) follows obviously. It is obvious that the right-hand side of the equation (D.2) is again continuously differentiable with respect to $i_t$, when $i_t \in \mathcal{I}$. Hence the function $L_t = \mathcal{F}(i_t)$ is twice continuously differentiable.
Since the function \( g(L_t) \) attains the maximum when \( L_t = \overline{L}_t \), it would thus not be plausible for the household to invest more than \( \overline{i}_t \). The \( \overline{i}_t \) is thus the higher bound of investment that a household can achieve in a given period. Because of the existence of the span-of-control parameter \( \nu \), there is always such a higher bound of investment, even when \( a_t = c_t \). However, in the latter case, the \( \overline{i}_t \) must be very small.

The discussion above presume that the household’s choice of consumption is smaller than the cash-on-hand, \( a_t \). However, it is feasible that the household choose to consume more than \( a_t \). When \( a_t < c_t \), \( f(i_t) \) is not monotonically increasing any more. It is instead first decreasing than increasing. Therefore, there exists minimal value of \( f(i_t) \) when \( a_t < c_t \). Such a minimal value can be used to define an “absolute higher bound” of consumption, \( \overline{c}_t \), as is shown in the next proposition.

**Proposition D.2** Given \( a_t > 0 \) and \( \eta_t \), there exists an “absolute higher bound” of consumption, \( \overline{c}_t \), that depends only on the \( a_t \) and \( \eta_t \):

\[
\overline{c}_t = a_t + (1 - \nu)\nu^{\frac{\nu}{1-\nu}} \left[ (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} g(\overline{L}_t) \right]^{\frac{1}{1-\nu}}.
\]  

(D.5)

When \( c_t \in (a_t, \overline{c}_t) \), there exists lower and upper bound of investment \( \widehat{i}_t \), \( \overline{i}_t \in (0, \infty) \), as well as a lower bound of leverage \( \widehat{L}_t \in (0, \overline{L}_t) \), such that \( L_t = F(i_t) \) is a function from \( \mathcal{I} = (\widehat{i}_t, \overline{i}_t) \) to \( \mathcal{L}^i = (\widehat{L}_t, \overline{L}_t) \). The function is decreasing on the interval \( (\widehat{i}_t, \overline{i}_t^\circ) \), and increasing on \( (\overline{i}_t^\circ, \overline{i}_t) \), where \( \overline{i}_t^\circ \) is given by

\[
\overline{i}_t^\circ = \frac{\nu}{1 - \nu} (c_t - a_t).
\]  

(D.6)

Furthermore, \( L_t \to \overline{L}_t \) when either \( i_t \to \widehat{i}_t \) or \( i_t \to \overline{i}_t \). Lastly, \( \widehat{i}_t, \widehat{L}_t, \overline{i}_t^\circ \to 0 \) when \( c_t \to a_t^+ \).

**Proof** When \( c_t > a_t \), we can arrange of formulation of the first-order derivative of \( f(i_t) \) as

\[
f'(i_t) = \frac{(1 - \nu) - \nu (c_t - a_t)i_t^{-1}}{(1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} i_t^{-\nu}}.
\]  

(D.7)

When \( c_t - a_t > 0 \), \( f'(i_t) \) is no longer always larger than 0 for any \( i_t > 0 \). We denote by \( i_t^\circ \) the root to the equation \( f'(i_t) = 0 \), and \( i_t^\circ \) can be found as

\[
i_t^\circ = \frac{\nu}{1 - \nu} (c_t - a_t).
\]

One can easily establish that \( f(i_t) \) is decreasing when \( i_t \leq i_t^\circ \) and increasing when \( i_t > i_t^\circ \). Remember that \( g(L_t) \) is increasing on the interval \( (0, \overline{L}_t) \). Thus consumption is only feasible if \( f(i_t^\circ) < g(\overline{L}_t) \). The
higher bound of consumption $\bar{c}_t$ is thus defined as the level of consumption that equates the inequality. Namely,

$$f \left( \frac{\nu}{1 - \nu} (\bar{c}_t - a_t) \right) = g(\bar{L}_t).$$

The root to the equation above is our formulation of $\bar{c}_t$ in the equation (D.5). When $c_t < \bar{c}_t$, the lower bound of the leverage $\bar{L}_t$ can be found by solving $f(i_t^\circ) = g(\bar{L}_t)$. Further, the lower and higher bound of investment $\hat{i}_t$ and $\tilde{i}_t$ are the two roots of $i_t$ to the equation $f(i_t) = g(\bar{L}_t)$. Since when $c_t > a_t$, $\lim_{i_t \to 0} f(i_t) = \lim_{i_t \to \infty} f(i_t) = \infty$; At the same time, $f(i_t)$ is decreasing on $(0, i_t^\circ]$ and increasing on $[i_t^\circ, \infty)$. Therefore, the condition that $c_t < \bar{c}_t$ would guarantee that the two roots of $i_t$ to the equation $f(i_t) = g(\bar{L}_t)$ are well-established.

By the monotonic property of the function $g(L_t)$, we can conclude that as long as $a_t < c_t < \bar{c}_t$, $L_t = \mathcal{F}(i_t)$ is a well-defined function from $\mathcal{I}$ to $\mathcal{L}^i$. Finally, concerning the limiting behaviour of $\hat{i}_t$, $\bar{L}_t$ and $i_t^\circ$ when $c_t \to a_t^+$. That $\lim_{c_t \to a_t^+} i_t^\circ = 0$ is easy to observe from the equation (D.6). To check for $\bar{L}_t$, we notice that

$$\lim_{c_t \to a_t^+} f(i_t^\circ) = \lim_{c_t \to a_t^+} f \left( \frac{\nu}{1 - \nu} (\bar{c}_t - a_t) \right) = \lim_{c_t \to a_t^+} \frac{1}{(1 - \nu)^{1-\nu} \nu^\nu} \cdot \frac{1}{(1 + r_{t+1}^c)} \exp \{ \eta_t \} \cdot (c_t - a_t)^{1-\nu} = 0,$$

and remember that $g(0) = 0$ and $g(\bullet)$ is continuous. Since given $a_t$ and $\eta_t$, for any $c_t > a_t$, $\bar{L}_t$ is the root to the equation $f(i_t^\circ) = g(\bar{L}_t)$, we can thus conclude that $\bar{L}_t \to 0$ when $c_t \to a_t^+$. The limiting behaviour of $\hat{i}_t$ can be established by observing that $0 < \hat{i}_t < i_t^\circ$.

If parameters are chosen appropriately, $\bar{c}_t$ is very close to $a_t$. In such case, it is very unlikely that the household may choose to consume more than she has. No matter what the relationship between $c_t$ and $a_t$ is, the function from investment to leverage is always well-defined on a meaningful range of investment. The first-order derivative of which is always given by the equation (D.2).

**D.1.2 The relationship between the investment and the saving**

We can now derive all other derivatives related to the household’s choice between investment and savings. This helps to consolidate the two choice variables $i_t$ and $s_t$ into one. First, we can show that for $i_t \in \mathcal{I}$, $s_t$ is a well-defined function of $i_t$, and we can characterize the first-order derivative of which analytically.

**Lemma D.3** Given $a_t > 0$, $\eta_t$ and $c_t \in (0, \bar{c}_t)$, $s_t$ is well-defined twice continuously differentiable function of $i_t$ for $i_t \in \mathcal{I}$. Moreover, the first-order derivative of $s_t$ with respect to $i_t$ is

$$\frac{\partial s_t}{\partial i_t} = -\nu(1 - \varphi)(1 + r_{t+1}^c) \exp \{ \eta_t \} \nu^{-1} L_t - \frac{(1 - \nu) + \nu(a_t - c_t)i_t^{-1}}{1 - \theta L_t^c(1 + \zeta)},$$

(D.8)
Proof From the Proposition D.1, if \( i_t \in \mathcal{I} \), then \( L_t = \mathcal{F}(i_t) \in \mathcal{L} \) (or \( \in \mathcal{L}^i \)) is well-defined. Further, based on the definition of the leverage we have

\[
s_t = -(1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} \tilde{i}_t L_t,
\]

hence the function from \( i_t \) to \( s_t \) is also well-defined. The first-order derivative of \( s_t \) with respect to \( i_t \) writes

\[
\frac{\partial s_t}{\partial i_t} = -\nu(1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} \tilde{i}_t^{-1} L_t - (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} \tilde{i}_t \cdot \frac{\partial L_t}{\partial i_t}.
\]

Take the representation of \( \frac{\partial L_t}{\partial i_t} \) (equation (D.2)) into the equation above and we obtain the equation (D.8). Further, since the equation (D.8) exists and is obviously continuously differentiable, thus \( s_t \) is twice continuously differentiable in \( i_t \). □

Next, we’ll see that the interest rate \( \pi_t \) is also a well-defined function of \( i_t \) on \( \mathcal{I} \). There is nice representation of the first-order derivative of \( \pi_t \) with respect to \( i_t \).

**Lemma D.4** Given \( a_t > 0 \), \( \eta_t \) and \( c_t \in (0, \widehat{c}_t) \), \( \pi_t \) is a well-defined twice continuously differentiable function of \( i_t \) for \( i_t \in \mathcal{I} \). Moreover, the first-order derivative of \( \pi_t \) with respect to \( i_t \) is

\[
\frac{\partial \pi_t}{\partial i_t} = -\frac{1}{s_t L_t} \cdot \frac{\Phi\left( \frac{\log((1+\pi_t) L_t) - \sigma^2/2}{\sigma} \right)}{1 - \Phi\left( \frac{\log((1+\pi_t) L_t) + \sigma^2/2}{\sigma} \right)} \cdot \frac{(1 - \nu) + \nu(a_t - c_t)\tilde{i}_t^{-1}}{1 - \theta L_t^c(1 + \zeta)}.
\]

(D.9)

If \( c_t \in (0, a_t] \), \( \pi_t \) is strictly increasing in \( \mathcal{I} \); while if \( c_t \in (a_t, \widehat{c}_t) \), \( \pi_t \) is first decreasing on \( (\widehat{i}_t, \tilde{i}_t^c] \) then increasing on \( (\tilde{i}_t^c, \tilde{i}_t) \).

**Proof** From the Proposition D.1, if \( i_t \in \mathcal{I} \), then \( L_t = \mathcal{F}(i_t) \in \mathcal{L} \) (on \( \in \mathcal{L}^i \)) is well-defined. It is strictly increasing on \( \mathcal{I} \) when \( c_t \in (0, a_t] \); By the Proposition D.2, when \( c_t \in (a_t, \widehat{c}_t) \), the function is first decreasing on \( (\widehat{i}_t, \tilde{i}_t^c] \) and then increasing on \( (\tilde{i}_t^c, \tilde{i}_t) \). From the Proposition 4.2, when \( L_t \in \mathcal{L}^i \subset \mathcal{L} \), \( \pi_t = \Pi(L_t) \in \mathcal{R} \) is also well-defined and strictly increasing. Therefore, a function from \( i_t \in \mathcal{I} \) to \( \pi_t \in \mathcal{R} \) is always well-defined, with the monotonic property follows that of the function \( \mathcal{F}(\bullet) \). The first-order derivative of \( \pi_t \) with respect to \( i_t \) can be calculated by the chain rule

\[
\frac{\partial \pi_t}{\partial i_t} = \frac{\partial \pi_t}{\partial L_t} \cdot \frac{\partial L_t}{\partial i_t},
\]

where the two derivatives on the right-hand side of the equation can be found in the equation (4.20) and (D.2), respectively. Moreover, since the equation (D.9) exists and is obviously continuously differentiable, thus \( \pi_t \) is twice continuously differentiable in \( i_t \). □
D.1.3 The relationship between the consumption and the saving

In this section, similar calculations are performed for the relationship between the consumption and the saving. First, given \( a_t \) and \( i_t \), the equation (D.1) defines a correspondence between the consumption and the leverage, \( L_t = J(c_t) \). Similar to the Proposition D.1 and D.2, the following proposition confirms that given \( a_t \) and \( c_t \), \( L_t = J(c_t) \) is a well-defined monotonically increasing function of \( c_t \) on some interval of \( c_t \), with a well-defined first-order derivative.

**Proposition D.5** Given \( a_t > 0 \) and \( \eta_t \), there exists an “absolute higher bound” of investment, \( \tilde{i}_t \), that depends only on the \( a_t \) and \( \eta_t \), such that investment must be less than \( \tilde{i}_t \). Given \( a_t > 0 \), \( \eta_t \) and \( i_t \in (0, \tilde{i}_t) \), there exists a lower bound of consumption \( \hat{c}_t = \max\{0, a_t - i_t\} \) and an upper bound of consumption \( \bar{c}_t \in (\hat{c}_t, \tilde{c}_t) \), as well as a lower bound of the leverage \( \hat{L} \in [0, L_t] \), such that \( L_t = J(c_t) \) is a one-to-one mapping from \( \mathcal{C} = (\hat{c}_t, \tilde{c}_t) \) to \( \mathcal{L}^c = (\hat{L}_t, L_t) \). The function \( L_t = J(c_t) \) is twice continuously differentiable and strictly increasing. Further, the derivative of the function \( L_t = J(c_t) \) is given by

\[
\frac{dL_t}{dc_t} = \frac{1}{(1 - \varphi)(1 + r_{t+1}^c)} \exp\{\eta_t\} i_t^{\nu} \cdot \frac{1}{1 - \theta J_t^{\alpha}(1 + \zeta)} > 0.
\]  

(D.10)

**Proof** The proof is similar to that of the Proposition D.1. We denote the left-hand side of the equation (D.1) as \( l(c_t) \), so that the equation (D.1) becomes \( l(c_t) = g(L_t) \). The function \( l(c_t) \) is linear in \( c_t \), with the first-order derivative

\[
l'(c_t) = \frac{1}{(1 - \varphi)(1 + r_{t+1}^c)} \exp\{\eta_t\} i_t^{\nu} > 0,
\]

hence if \( i_t \neq 0 \), \( l'(c_t) \) is monotonically increasing in \( c_t \). Further,

\[
\lim_{c_t \to 0} l(c_t) = \frac{i_t - a_t}{(1 - \varphi)(1 + r_{t+1}^c)} \exp\{\eta_t\} i_t^{\nu} \quad \text{and} \quad \lim_{c_t \to \infty} l(c_t) = +\infty.
\]  

(D.11)

The properties of \( g(L_t) \) is the same as before. We discuss the solution of \( L_t \) in the following case:

**Case of** \( 0 < i_t \leq a_t \). Define \( \tilde{c}_t \), such that \( l(\tilde{c}_t) = 0 \), namely, \( \tilde{c}_t = a_t - i_t \). Further, define \( \bar{c}_t \) such that \( \ell(\bar{c}_t) = g(L_t) \), namely,

\[
\bar{c}_t = (a_t - i_t) + (1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t\} i_t^{\nu} g(L_t).
\]  

(D.12)

We let \( \hat{L}_t := 0 \). Similar to the logic in the proof of the Proposition D.1, the function \( L_t = J(c_t) \) is a well-defined strictly increasing bijective function from \( \mathcal{C} = (\hat{c}_t, \tilde{c}_t) \) to \( \mathcal{L}^c = (\hat{L}_t, L_t) \). The function is also twice continuously differentiable.
Case of $a_t < i_t$. Define $\tilde{c}_t := 0$. Further, define $\hat{L}_t$ such that $\lim_{c_t \to 0} l(c_t) = g(\hat{L}_t)$, namely, $\hat{L}_t$ solves the following non-linear equation

$$\frac{i_t - a_t}{(1 - \varphi)(1 + r^c_{t+1})\exp\{\eta_t\}i^\nu_t} = \left[1 - \theta \hat{L}_t^\nu\right] \hat{L}_t. \quad (D.13)$$

At the same time, the definition of the $\tilde{c}_t$ is the same as the equation (D.12). Since the upper bound of consumption cannot drop below or equal to 0, investment must satisfy $g(\hat{L}_t) < g(\tilde{L}_t)$. Therefore, the “absolute higher bound” of investment $\tilde{i}_t$ can be defined as the level of investment that equates $g(\hat{L}_t)$ to $g(\tilde{L}_t)$, which depends only on $a_t$ and $\eta_t$. More specifically, $\tilde{i}_t$ can be found by solving the following nonlinear equation

$$\frac{\tilde{i}_t - a_t}{(1 - \varphi)(1 + r^c_{t+1})\exp\{\eta_t\}i^\nu_t} = \left[1 - \theta \tilde{L}_t^\nu\right] \tilde{L}_t. \quad (D.14)$$

To prevent consumption dropping below (or equal to) zero, investment must satisfy $i_t < \tilde{i}_t$. If $i_t < \tilde{i}_t$, the function $L_t = J(c_t)$ is a well-defined strictly increasing bijective function from $\mathcal{C} = (\hat{c}_t, \tilde{c}_t)$ to $\mathcal{L} = (\hat{L}_t, \tilde{L}_t)$. The function is also twice continuously differentiable.

When $c_t \in \mathcal{C}$, the first-order derivative ($dL_t/dc_t$) can be found by applying the total differential on the equation (D.1), in a similar way as what in the proof of the Proposition D.1:

$$0 = l'(c_t) \, dc_t - g'(L_t) \, dL_t. \quad \blacksquare$$

Next, similar to he Lemma D.3 and D.4, the first-order derivative of $s_t$ and $\pi_t$ with respect to $c_t$ can also by analytically characterized.

**Lemma D.6** Given $a_t > 0$, $\eta_t$ and $i_t \in (0, \tilde{i}_t)$, $s_t$ is a well-defined twice continuously differentiable and strictly decreasing in $c_t$ for $c_t \in \mathcal{C}$. Moreover, the first-order derivative of $s_t$ with respect to $c_t$ is

$$\frac{\partial s_t}{\partial c_t} = -\frac{1}{1 - \theta L_t^\nu(1 + \zeta)} < 0. \quad (D.15)$$

**Proof** From the Proposition D.5, if $c_t \in \mathcal{C}$, then $L_t = J(c_t) \in \mathcal{L}^c$ is well-defined. Further, based on the definition of the leverage we have

$$s_t = -(1 - \varphi)(1 + r^c_{t+1})\exp\{\eta_t\}i^\nu_t L_t,$$

hence the function from $c_t$ to $s_t$ is also well-defined. The first-order derivative of $s_t$ with respect to $c_t$ writes

$$\frac{\partial s_t}{\partial c_t} = -(1 - \varphi)(1 + r^c_{t+1})\exp\{\eta_t\}i^\nu_t \cdot \frac{\partial L_t}{\partial c_t}. \quad 74$$
Take the representation of the $\frac{\partial L_t}{\partial i_t}$ (equation (D.10)) into the equation above and we obtain the equation (D.15). Because when $c_t \in C$, we have $L_t \in L^c$ and $g'(L_t) = 1 - \theta L_t^\zeta(1 + \zeta) > 0$. Therefore, the first-order derivative in (D.15) is always larger than zero. Furthermore, since the equation (D.15) exists and is obviously continuously differentiable, $s_t$ is twice continuously differentiable in $c_t$. ■

**Lemma D.7** Given $a_t > 0$, $\eta_t$ and $i_t \in (0, \tilde{i}_t)$, $\pi_t$ is a well-defined twice continuously differentiable and strictly increasing in $c_t$ for $c_t \in C$. Moreover, the first-order derivative of $\pi_t$ with respect to $c_t$ is

$$
\frac{\partial \pi_t}{\partial c_t} = - \frac{1}{s_t L_t} \cdot \Phi \left( \frac{\log \left( \frac{(1+\pi_t)L_t}{\sigma} \right)}{\sigma} - \frac{\sigma^2}{2} \right) \cdot \frac{1}{1 - \Phi \left( \frac{\log \left( \frac{(1+\pi_t)L_t}{\sigma} \right)}{\sigma} + \frac{\sigma^2}{2} \right)} > 0.
$$

**Proof** From the Proposition D.5, if $c_t \in C$, $L_t = F(c_t) \in L^c$ is well-defined and strictly increasing. From the Proposition 4.2, when $L_t \in L^c \subset L$, $\pi_t = \Pi(L_t) \in R$ is also well-defined and strictly increasing. Therefore, a function from $c_t \in C$ to $\pi_t \in R$ is always well-defined and strictly increasing. The first-order derivative of $\pi_t$ with respect to $c_t$ can be calculated by the chain rule

$$
\frac{\partial \pi_t}{\partial c_t} = \frac{\partial \pi_t}{\partial L_t} \cdot \frac{\partial L_t}{\partial c_t},
$$

where the two derivatives on the right-hand side of the equation can be found in the equation (4.20) and (D.10), respectively. Moreover, since the equation (D.16) exists and obviously continuously differentiable, thus $\pi_t$ is twice continuously differentiable in $c_t$. ■

### D.2 Next-period State

Household’s next-period cash-on-hand can be represented as a function $c_t$, $i_t$ and $\varepsilon_t$, as well as $a_t$ and $\eta_t$:

$$
a_{t+1} = A(c_t, i_t, \varepsilon_t; a_t, \eta_t)
$$

$$
= \begin{cases} 
  w_{t+1} + (1 + r^c_{t+1}) \exp\{\eta_t + \varepsilon_t\} i_t^\nu + (1 + r^f_t)(a_t - c_t - i_t), & \text{if } s_t \geq 0; \\
  w_{t+1} + (1 + r^c_{t+1}) \exp\{\eta_t + \varepsilon_t\} i_t^\nu + (1 + \pi_t)s_t, & \text{if } s_t < 0 \text{ and solvent}; \\
  w_{t+1}, & \text{if } s_t < 0 \text{ and insolvent}.
\end{cases}
$$

(D.17)

In this section, we first derive the first-order derivative of $A(\bullet)$ with respect to $c_t$ and $i_t$, given $a_t > 0$ and $\eta_t$. It is crucial to divide our discussion into two cases, one is when $c_t + i_t \leq a_t$, and another $c_t + i_t > a_t$. In the former case, there is no borrowing, and hence the transaction cost $\kappa_t$ is
zero. The household only faces the risk-free interest rate if she doesn’t borrow. In the latter case, the borrowing interest rate depends on the amount of borrowing. The two cases have two different formulations of the first-order partial derivatives w.r.t. $c_t$ and $i_t$. But thanks to the technical term $L_t^\zeta$ in our formulation of the transaction cost, the partial derivatives are smooth when $(c_t + i_t)$ passes through $a_t$. In the end we are able to show that $A(\bullet)$ is continuously differentiable in both argument of $c_t$ and $i_t$.

We will also derive the first-order derivative of $A(\bullet)$ with respect to $a_t$, given $c_t$, $i_t$ and $\eta_t$. This partial derivative enable us to apply envelope condition to back out the formulation of $\frac{\partial V}{\partial a}$, which shows up in the first-order conditions of the consumption and investment.

**D.2.1 Next-period Cash-on-hand with respect to choices**

**Case of** $c_t + i_t \leq a_t$. In this case, the equation (D.17) becomes

$$A(c_t, i_t, \varepsilon_t; a_t, \eta_t) = (1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^\nu + (1 + r_t^f)(a_t - c_t - i_t) + w_{t+1}. \quad (D.18)$$

The first-order partial derivative of $A(i_t, \varepsilon_t)$ with respect to $c_t$ and $i_t$ are

$$\frac{\partial A(\bullet)}{\partial c_t} = -(1 + r_t^f); \quad (D.19)$$

$$\frac{\partial A(\bullet)}{\partial i_t} = \nu(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^{\nu-1} - (1 + r_t^f). \quad (D.20)$$

**Case of** $c_t + i_t > a_t$. Household is insolvent, namely $a_{t+1} = w_{t+1}$, when

$$(1 - \varphi)(1 + r_{t+1}^c) \exp\{\eta_t + \varepsilon_t\} i_t^\nu + (1 + \pi_t) s_t \leq 0 \iff \varepsilon_t \leq \log((1 + \pi_t)L_t). \quad (D.21)$$

Based on the discussion in the last section, when $c_t \in C$ and $i_t \in I$, both $\pi_t$ and $L_t$ can be represented as well-defined functions of $c_t$ and $i_t$. We thus adopt the notation $E(c_t, i_t) := \log((1 + \pi_t)L_t)$.

When $\varepsilon_t \leq E(c_t, i_t)$, $A(\bullet) \equiv w_{t+1}$. In this case, $\frac{\partial A(\bullet)}{\partial c_t} = \frac{\partial A(\bullet)}{\partial i_t} \equiv 0$. When $\varepsilon_t > E(c_t, i_t)$, using the Lemma D.3, Lemma D.4, Lemma D.6 and Lemma D.7, it can be shown that

$$\frac{\partial A(\bullet)}{\partial c_t} = (1 + \pi_t) \frac{\partial s_t}{\partial c_t} + s_t \frac{\partial \pi_t}{\partial c_t}$$

$$= -\frac{1}{1 - \theta L_t^\zeta (1 + \zeta)} \left[ (1 + \pi_t) + \frac{1}{L_t} \cdot \Phi \left( \frac{\log((1+\pi_t)L_t) - \sigma^2/2}{\sigma} \right) - 1 \right]. \quad (D.22)$$
\[
\frac{\partial A(\bullet)}{\partial t_t} = \nu(1 + r^c_{t+1}) \exp\{\eta_t + \varepsilon_t\}\nu^{-1} + (1 + \pi_t) \frac{\partial s_t}{\partial t_t} + s_t \frac{\partial \pi_t}{\partial t_t}
\]

\[
= \nu(1 + r^c_{t+1}) \exp\{\eta_t + \varepsilon_t\}\nu^{-1} - \nu(1 - \varphi)(1 + r^c_{t+1}) \exp\{\eta_t\}\nu^{-1}(1 + \pi_t)L_t
\]  

(D.23)

\[
- \frac{(1 - \nu) + \nu(a_t - c_t)\nu^{-1}}{1 - \theta L^c_t(1 + \zeta)} \left[ (1 + \pi_t) + \frac{1}{L_t} \cdot \Phi \left( \frac{\log \left( (1 + \pi_t)L_t - \sigma^2/2 \right)}{\sigma} \right) \right].
\]

Applying the equation (4.18) and the definition of the leverage \(L_t\), we are able to show that

\[
\frac{\partial A(\bullet)}{\partial c_t} = -\frac{1}{1 - \theta L^c_t(1 + \zeta)} \cdot \frac{1 + r^f_t}{1 - \Phi \left( \frac{\log \left( (1 + \pi_t)L_t + \sigma^2/2 \right)}{\sigma} \right)};
\]  

(D.24)

\[
\frac{\partial A(\bullet)}{\partial i_t} = \nu(1 + r^c_{t+1}) \exp\{\eta_t + \varepsilon_t\}\nu^{-1} + \nu(1 + \pi_t) \frac{s_t}{i_t}
\]

\[
- \frac{(1 - \nu) + \nu(a_t - c_t)\nu^{-1}}{1 - \theta L^c_t(1 + \zeta)} \cdot \frac{1 + r^f_t}{1 - \Phi \left( \frac{\log \left( (1 + \pi_t)L_t + \sigma^2/2 \right)}{\sigma} \right)}.
\]  

(D.25)

**Continuous Differentiability of the function \(A(\bullet)\).** We now show that given \(a_t > 0\) and \(\eta_t\), the next-period state \(A(c_t, i_t, \varepsilon_t; a_t, \eta_t)\) is a continuously differentiable function in \(c_t\) and \(i_t\), for \(c_t \in C^+ := (0, \bar{c}_t)\) and \(i_t \in I^+ := (\hat{i}_t, \bar{i}_t)\), where \(\hat{i}_t = 0\) if \(c_t \leq a_t\). The range \(C^+\) and \(I^+\) are ranges of \(c_t\) and \(i_t\) on which the optimization is performed. Note that \(\bar{c}_t\) depends on the household’s choice of \(i_t\), while \(\bar{i}_t\) depends on her choice of \(c_t\). If \(c_t > a_t\), \(\bar{i}_t\) also depends on \(c_t\). The crucial thing is that, when household passes through the point where she has to borrow, her next-period period state should be continuously differentiable in the two choice variables. This is obvious, since if \(c_t + i_t \to a^+_t\), \(s_t \to 0\), \(L_t \to 0\) and \(\mathcal{E}(c_t, i_t) \to -\infty\). The \(\Phi(\bullet)\)-term in the equation (D.24) and (D.25) approaches to zero as a result. Thanks to the technical term \(L_t^c\), when \(L_t \to 0\), \((1 - \theta L^c_t(1 + \zeta))^{-1} \to 1\). Therefore, the equation (D.24) and (D.25) converges to (D.19) and (D.20) when \(c_t + i_t \to a^+_t\). This is concluded in the following proposition.

**Proposition D.8** Given \(a_t > 0\) and \(\eta_t\) and if \(c_t \in C^+\) and \(i_t \in I^+\), \(A(c_t, i_t, \varepsilon_t; a_t, \eta_t)\) is a continuously differentiable function in \(c_t\) and \(i_t\).
Next-period Cash-on-hand with respect to current cash-on-hand

Given $\eta_t$ and the choice of $c_t \in \mathcal{C}^+$ and $i_t \in \mathcal{I}^+$, if $c_t + i_t > a_t$, the equation (D.1) defines a function from $a_t$ to $L_t$. The first-order derivative of $L_t$ with respect to $a_t$ writes

$$\frac{dL_t}{da_t} = -\frac{1}{(1 - \varphi)(1 + r_{t+1}^c)} \exp\{\eta_t\} i_t \cdot \frac{1}{1 - \theta L_t \zeta^2 (1 + \zeta)} < 0.$$ (D.26)

Under the case of $a_t < c_t + i_t$, following the same methods in the proof of Lemma D.3, Lemma D.4, Lemma D.6 and Lemma D.7, the first-order derivative of $s_t$ with respect to $a_t$ writes

$$\frac{\partial s_t}{\partial a_t} = \frac{1}{1 - \theta L_t \zeta^2 (1 + \zeta)} > 0;$$ (D.27)

and the first-order derivative of $\pi_t$ with respect to $a_t$ writes

$$\frac{\partial \pi_t}{\partial a_t} = \frac{1}{s_t L_t} \cdot \Phi\left(\frac{\log\left(\frac{(1 + \pi_t) L_t - \sigma^2 / 2}{\sigma}\right)}{\sigma}\right) \cdot \frac{1}{1 - \theta L_t \zeta^2 (1 + \zeta)} < 0.$$ (D.28)

We are now ready to derive the first-order derivative of $A(\bullet)$ with respect to $a_t$. The derivation is again divided into two cases. If $c_t + i_t \leq a_t$,

$$\frac{\partial A(\bullet)}{\partial a_t} = (1 + r_t^f);$$ (D.29)

while if $c_t + i_t > a_t$, $\frac{\partial A(\bullet)}{\partial a_t} \equiv 0$ when $\varepsilon_t \leq \mathcal{E}(c_t, i_t)$; When $\varepsilon_t > \mathcal{E}(c_t, i_t)$,

$$\frac{\partial A(\bullet)}{\partial a_t} = (1 + \pi_t) \frac{\partial s_t}{\partial a_t} + s_t \frac{\partial \pi_t}{\partial a_t}$$

$$= \frac{1}{1 - \theta L_t \zeta^2 (1 + \zeta)} \left[ (1 + \pi_t) + \frac{1}{L_t} \cdot \Phi\left(\frac{\log\left(\frac{(1 + \pi_t) L_t - \sigma^2 / 2}{\sigma}\right)}{\sigma}\right) \right]$$

$$= \frac{1}{1 - \theta L_t \zeta^2 (1 + \zeta)} \cdot \frac{1 + r_t^f}{1 - \Phi\left(\frac{\log\left(\frac{(1 + \pi_t) L_t + \sigma^2 / 2}{\sigma}\right)}{\sigma}\right)}.$$ (D.30)

One notices that $\frac{\partial A(\bullet)}{\partial a_t} = -\frac{\partial A(\bullet)}{\partial c_t}$. Following the same logic as the Proposition D.8, $A(\bullet)$ is continuously differentiable in $a_t$. 78
D.3 Optimal Investment and Consumption

We are now ready to obtain the first-order conditions of the household’s optimization problem. The purpose is to obtain a well-defined non-linear equation systems of \( c_t \) and \( i_t \), to be solved numerically for the optimal decisions of household. Following the convention of the literature, we drop the subscript \( t \), and we use the prime-notation to represent the variables in the period \((t + 1)\).

D.3.1 Household’s problem revisited

Recall that the recursive representation of the household’s problem as follows:

\[
V(a, \eta; G) = \max_{c, i} \left\{ (1 - \beta) c^{1 - \mu} + \beta \left( \lambda_{\eta H} \int_{\varepsilon} V(a', \eta; G')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} V(a', 0; G')^{1 - \gamma} f(\varepsilon) d\varepsilon \right) \right\}^{\frac{1 - \mu}{1 - \gamma}} \frac{1}{1 - \mu},
\]

s.t. \( a' = A(c, i, \varepsilon; a, \eta) \),
\( c \in C^+, \ i \in I^+ \)

Remember from the Proposition D.8, the function \( A(c, i, \varepsilon; a, \eta) \) is a continuously differentiable function in \( c \in C^+ \) and \( i \in I^+ \). Therefore, we can derive the first-order conditions for \( c \) and \( i \). We further define

\[
\tilde{a}(c, i; \eta) := \varphi(1 + r^{\varepsilon'}) \exp\{E(i, c) + \eta\} i^{\nu} + w
\]
as the bankruptcy deadweight loss.

D.3.2 First-order conditions of investment and consumption

If \( \gamma > 1 \) and \( \mu > 1 \), after the consumption is chosen, the household only needs to choose the investment \( i^* \), such that

\[
i^* = \arg \min_{i \in I^+} \left\{ \lambda_{\eta H} \int_{\varepsilon} V(a', \eta; G')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} V(a', 0; G')^{1 - \gamma} f(\varepsilon) d\varepsilon \right\}.
\]
The next proposition provides the first-order condition of investment, if the optimal investment exists.

Proposition D.9 (First-order Necessary Condition for Investment) Suppose that the value function \( V(a, \eta; G) \) is continuously differentiable in \( a \). Given \( a > 0 \) and \( \eta \), and for given \( c \in (0, \tilde{c}) \), if \( i^* \in I^+ \) exists, \( i^* \) must solve

\[\text{Proposition D.9 (First-order Necessary Condition for Investment) Suppose that the value function } V(a, \eta; G) \text{ is continuously differentiable in } a. \text{ Given } a > 0 \text{ and } \eta, \text{ and for given } c \in (0, \tilde{c}), \text{ if } i^* \in I^+ \text{ exists, } i^* \text{ must solve}\]

\[\text{In the following, to keep the formulation succinct, we denote the } A(c, i, \varepsilon; a, \eta) \text{ as } A(c, i, \varepsilon).\]
\[ 0 = (1 - \gamma) \int_{E(c,i^*)}^{\infty} D(c,i^*,\varepsilon) \frac{\partial A(c,i^*,\varepsilon)}{\partial i^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right) \, d\varepsilon \]

\[ + \phi \left( \frac{E(c,i^*) + \sigma^2/2}{\sigma} \right) \cdot \frac{\partial E(c,i^*)}{\partial i^*} \cdot C(c,i^*), \tag{D.31} \]

where \( E(c,i^*) = -\infty \) if \( c + i^* \leq a \). \( \phi(\bullet) \) is the probability density function of the standard normal distribution. Moreover,

\[ D(c,i^*,\varepsilon) = \lambda_{\eta H} V(A(c,i^*,\varepsilon),\eta; G')^{-\gamma} \cdot \frac{\partial V(A(c,i^*,\varepsilon),\eta; G')}{\partial A(c,i^*,\varepsilon)} + \lambda_{\eta L} V(A(c,i^*,\varepsilon),0; G')^{-\gamma} \cdot \frac{\partial V(A(c,i^*,\varepsilon),0; G')}{\partial A(c,i^*,\varepsilon)}. \tag{D.32} \]

and \( C(\bullet) \) can be deemed as the “marginal utility cost” of bankruptcy, which is

\[ C(c,i^*) = \lambda_{\eta H} \left[ V(w,\eta; G')^{1-\gamma} - V(\bar{a},\eta; G')^{1-\gamma} \right] + \lambda_{\eta L} \left[ V(w,0; G')^{1-\gamma} - V(\bar{a},0; G')^{1-\gamma} \right]. \tag{D.33} \]

**Proof** For given \( a > 0, \eta \) and \( c \in (0,\bar{c}) \), define \( H(i;\eta) \) to be

\[ H(i;\eta) = \mathbb{E}_\varepsilon \left[ V(A(c,i,\varepsilon),\eta; G')^{1-\gamma} \right] = \int_{\varepsilon} V(A(c,i,\varepsilon),\eta; G')^{1-\gamma} f(\varepsilon) \, d\varepsilon. \]

We can thus equivalently represent the \( i^* \) as

\[ i^* = \arg \min_{i \in I^+} \left\{ \lambda_{\eta H} H(i;\eta) + \lambda_{\eta L} H(i;0) \right\}. \tag{D.34} \]

When \( i \in I^+ \), the first-order derivative of \( H(\bullet) \) with respect to \( i \) is

\[ \frac{\partial H(\bullet)}{\partial i} = \frac{\partial}{\partial i} \left[ \int_{-\infty}^{E(c,i)} V(w,\eta; G')^{1-\gamma} f(\varepsilon) \, d\varepsilon + \int_{E(c,i)}^{\infty} V(A(c,i,\varepsilon),\eta; G')^{1-\gamma} f(\varepsilon) \, d\varepsilon \right]. \tag{D.35} \]

By the Leibniz rule, we have

\[ \frac{\partial}{\partial i} \int_{-\infty}^{E(c,i)} V(w,\eta; G')^{1-\gamma} f(\varepsilon) \, d\varepsilon = V(w,\eta; G')^{1-\gamma} f(E(c,i)) \cdot \frac{\partial E(c,i)}{\partial i}; \tag{D.36} \]

and since \( A(c,i,E(c,i)) = \bar{a}(c,i) \),

\[ \frac{\partial}{\partial i} \int_{E(c,i)}^{\infty} V(A(c,i,\varepsilon),\eta; G')^{1-\gamma} f(\varepsilon) \, d\varepsilon = -V(\bar{a},\eta; G')^{1-\gamma} f(E(c,i)) \cdot \frac{\partial E(c,i)}{\partial i} \]

\[ + (1 - \gamma) \int_{E(c,i)}^{\infty} \left[ V(A(c,i,\varepsilon),\eta; G')^{\gamma} \cdot \frac{\partial V(A(c,i,\varepsilon),\eta; G')}{\partial A(c,i,\varepsilon)} \cdot \frac{\partial A(c,i,\varepsilon)}{\partial i} \right] f(\varepsilon) \, d\varepsilon. \tag{D.37} \]
Take the equation (D.36) and (D.37) to the equation (D.35) and we have
\[
\frac{\partial H(\bullet)}{\partial i} = (1 - \gamma) \int_{\infty} \left[ \nabla(A(c, i, \varepsilon), \eta; \theta') \right] \cdot \frac{\partial A(c, i, \varepsilon)}{\partial i} \cdot f(\varepsilon) \, d\varepsilon
\]
\[+ f(E(c, i)) \cdot \frac{\partial E(c, i)}{\partial i} \cdot \left[ (1 - \gamma) \nabla(V(\eta, \theta')) - (1 - \gamma) \nabla(V(\eta, \theta')) \right].
\]

By the Proposition D.8, the function \( A(\bullet) \) is continuously differentiable in \( i \in \mathbb{I}^+ \). Therefore, if the value function \( V(\bullet) \) is continuously differentiable in \( a \), the first-order derivative (D.38) is well defined.

By the equation (D.34), if \( i^* \in \mathbb{I}^+ \) exists, it must solve
\[
\lambda_{\eta H} \frac{\partial H(i^*; a)}{\partial i^*} + \lambda_{\eta L} \frac{\partial H(0; i^*)}{\partial i^*} = 0.
\]

Taking the representation of (D.38) to the equation above and we obtain
\[
0 = (1 - \gamma) \int_{E(c, i)} \mathbb{D}(c, i, \varepsilon) \frac{\partial A(c, i, \varepsilon)}{\partial i} f(\varepsilon) \, d\varepsilon + f(E(c, i)) \cdot \frac{\partial E(c, i)}{\partial i} \cdot \mathbb{C}(c, i^*).
\]

Since \( \varepsilon \sim \mathcal{N}(-\sigma^2/2, \sigma^2) \), hence \( f(\varepsilon) = \frac{1}{\sigma} \phi(\varepsilon + \sigma^2/2) \). Take this to the equation (D.39), we obtain the equation (D.31).

To put the first-order condition into numerical calculation, one needs to obtain the explicit form of the second term of the equation (D.31). To obtain the partial derivative of \( E(\bullet) \) when \( c + i > a \), we perform the following calculation and apply the Proposition D.1 and the Lemma D.4:
\[
\frac{\partial E(c, i)}{\partial i} = \frac{\partial}{\partial i} \log ((1 + \pi)L) = \frac{1}{(1 + \pi)L} \left[ (1 + \pi) \frac{\partial L}{\partial i} + L \frac{\partial \pi}{\partial i} \right]
\]
\[= \frac{1}{L} \cdot \frac{\partial L}{\partial i} + \frac{1}{1 + \pi} \cdot \frac{\partial \pi}{\partial i}
\]
\[= -\frac{1}{s} \cdot \frac{(1 - \nu) + \nu(a - c)i^{-1}}{g'(L)}
\]
\[= -\frac{1}{s} \cdot \frac{(1 - \nu) + \nu(a - c)i^{-1}}{g'(L)} \cdot \frac{1}{(1 + \pi)L} \cdot \Phi\left( \frac{\log (1 + \pi) - \sigma^2/2}{\sigma} \right)
\]
\[= -\frac{1}{s} \cdot \frac{(1 - \nu) + \nu(a - c)i^{-1}}{g'(L)} \cdot \frac{1}{(1 + \pi)L} \cdot \left[ (1 + \pi) + \frac{1}{L} \cdot \Phi\left( \frac{\log ((1 + \pi)L - \sigma^2/2)}{\sigma} \right) \right],
\]
\[\quad \text{D.40}
\]
where \( g'(L) = 1 - \theta L \zeta (1 + \zeta) \). Applying the non-linear equation for interest rate (Proposition 4.1) we obtain that

\[
\frac{\partial E(c, i)}{\partial i} = \frac{(1 - \nu) + \nu(a - c)i^{-1}}{(1 + \pi)s g'(L)} \cdot \frac{1 + r^f}{1 - \Phi \left( \frac{\log(1 + \pi)L + \sigma^2/2}{\sigma} \right)}.
\] (D.41)

Taking the equation (D.41) to the equation (D.31) and we obtain

\[
0 = \int_0^\infty \mathbb{D}(c, i^*, \varepsilon) \frac{\partial A(c, i^*, \varepsilon)}{\partial i^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right) d\varepsilon

- \frac{1 + r^f}{1 - \gamma} \cdot \frac{(1 - \nu) + \nu(a - c)i^{-1}}{s(1 + \pi)g'(L)} \cdot \phi \left( \frac{\varepsilon(c, i^*) + \sigma^2/2}{\sigma} \right) \cdot C(c, i^*);
\] (D.42)

or can we write

\[
0 = \int_0^\infty \mathbb{D}(c, i^*, \varepsilon) \frac{\partial A(c, i^*, \varepsilon)}{\partial i^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right) d\varepsilon

- \frac{1 + r^f}{1 - \gamma} \cdot \frac{(1 - \nu) + \nu(a - c)i^{-1}}{s(1 + \pi)g'(L)} \cdot m \left( \frac{-\varepsilon(c, i^*) + \sigma^2/2}{\sigma} \right) \cdot C(c, i^*);
\] (D.43)

where \( m(\bullet) = \phi(\bullet)/\Phi(\bullet) \) is the inverse Mills ratio. When \( c + i^* \leq a \), the second term of the equation (D.42) and (D.43) vanishes, and the first-order condition becomes

\[
0 = \mathbb{D}(c, i^*, \varepsilon) \frac{\partial A(c, i^*, \varepsilon)}{\partial i^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right).
\]

We now derive the first-order condition of consumption. If \( \gamma > 1 \) and \( \mu > 1 \), after the investment is chosen, the household chooses the consumption \( c^* \), such that

\[
c^* = \arg\min_{c \in C^+} \left\{ (1 - \beta)c^{1-\mu} + \beta \left( \lambda_{\eta H} \int_\varepsilon \mathbb{V}(a', \eta; \mathcal{G}')^{1-\gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_\varepsilon \mathbb{V}(a', 0; \mathcal{G}')^{1-\gamma} f(\varepsilon) d\varepsilon \right)^{\frac{1}{1-\gamma}} \right\}.
\]

In a similar manner as is in the proof of the Proposition D.10, we can derive the first-order necessary condition for consumption. The next proposition provides the formulation of the first-order condition, if the optimal consumption exists in \( C^+ \).

**Proposition D.10** (First-order Necessary Condition for Consumption) Suppose that the value function \( \mathbb{V}(a, \eta; \mathcal{G}) \) is continuously differentiable in \( a \). Given \( a > 0 \) and \( \eta \), and for given \( i \in (0, i^*) \), if \( c^* \in C^+ \) exists, \( c^* \) must solve
\[0 = \sigma (1 - \beta) (c^*)^{-\mu} + \frac{\beta}{1 - \gamma} \cdot \mathbb{CE} \left( \mathbb{V}(a', \eta'; \mathbb{G}') \right)^{\gamma - \mu}\]

\[
\left(1 - \gamma \right) \int_{\mathbb{E}(c^* i)}^\infty \mathbb{D}(c^* i \varepsilon) \frac{\partial \mathcal{A}(c^* i \varepsilon)}{\partial c^*} \phi \left( \varepsilon + \frac{\sigma^2 / 2}{\sigma} \right) d\varepsilon + \phi \left( \frac{\mathcal{E}(c^* i) + \sigma^2 / 2}{\sigma} \right) \frac{\partial \mathcal{E}(c^* i)}{\partial c^*} \cdot \mathbb{C}(c^* i)\]

where \(\mathbb{CE}(\bullet)\) is the certainty equivalence of the future value, and

\[
\mathbb{CE} \left( \mathbb{V}(a', \eta'; \mathbb{G}') \right) = \left[ \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon \right]^{\frac{1}{1 - \gamma}},
\]

and \(\mathbb{D}(c^* i \varepsilon)\) is similar as (D.32) and is

\[
\mathbb{D}(c^* i \varepsilon) = \lambda_{\eta H} \mathbb{V}(\mathcal{A}(c^* i \varepsilon), \eta; \mathbb{G}')^{\gamma} \cdot \frac{\partial \mathbb{V}(\mathcal{A}(c^* i \varepsilon), \eta; \mathbb{G}')}{\partial \mathcal{A}(c^* i \varepsilon)} + \lambda_{\eta L} \mathbb{V}(\mathcal{A}(c^* i \varepsilon), 0; \mathbb{G}')^{\gamma} \cdot \frac{\partial \mathbb{V}(\mathcal{A}(c^* i \varepsilon), 0; \mathbb{G}')}{\partial \mathcal{A}(c^* i \varepsilon)}.
\]

Note that we set \(\mathcal{E}(c^* i) = -\infty\) if \(c^* i \leq a\). Note that the partial term \(\frac{\partial \mathcal{A}(\bullet)}{\partial c}\) does not depend on the random term \(\varepsilon\).

**Proof** The first-order condition writes

\[
0 = (1 - \mu)(1 - \beta)c^{-\mu} + \beta \cdot \frac{1 - \mu}{1 - \gamma} \cdot \mathbb{CE} \left( \mathbb{V}(a', \eta'; \mathbb{G}') \right)^{\gamma - \mu} \cdot \frac{\partial}{\partial c} \left[ \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon \right],
\]

or

\[
0 = (1 - \beta)c^{-\mu} + \frac{\beta}{1 - \gamma} \cdot \mathbb{CE} \left( \mathbb{V}(a', \eta'; \mathbb{G}') \right)^{\gamma - \mu} \cdot \frac{\partial}{\partial c} \left[ \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon \right].
\]

Using the same method as the proof of the Proposition D.9., we have

\[
\frac{\partial}{\partial c} \left[ \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathbb{G}')^{1 - \gamma} f(\varepsilon) d\varepsilon \right] = (1 - \gamma) \int_{\mathbb{E}(c i)}^\infty \mathbb{D}(c i \varepsilon) \frac{\partial \mathcal{A}(c i \varepsilon)}{\partial c} f(\varepsilon) d\varepsilon + f(\mathcal{E}(c i)) \cdot \frac{\partial \mathcal{E}(c i)}{\partial c} \cdot \mathbb{C}(c i),
\]

which is taken to the equation (D.44), and we obtain the equation (D.44).
We need again to determine an explicit form of the first-order condition above. By Proposition D.1 and Lemma D.4, when \( c + i > a \), we see that

\[
\frac{\partial E(c, i)}{\partial c} = \frac{\partial }{\partial c} \log \left( (1 + \pi)L \right) = \frac{1}{(1 + \pi)L} \left[ (1 + \pi) \frac{\partial L}{\partial c} + L \frac{\partial \pi}{\partial c} \right]
\]

\[
= \frac{1}{L} \frac{\partial L}{\partial c} + \frac{1}{1 + \pi} \frac{\partial \pi}{\partial c}
\]

\[
= -\frac{1}{s g'(L)} - \frac{1}{s g'(L)} \cdot \frac{1}{(1 + \pi)L} \cdot \frac{\Phi \left( \frac{\log \left( (1 + \pi)L \right) - \sigma^2/2}{\sigma} \right)}{1 - \Phi \left( \frac{\log \left( (1 + \pi)L \right) + \sigma^2/2}{\sigma} \right)}
\]

(D.47)

Take the equation (D.47) into the FOC (D.44) and we obtain

\[
0 = \sigma (1 - \beta) \left( c^* \right)^{-\mu} + \beta \mathbb{E} \left( \mathcal{V}(a', \eta'; \mathcal{G}') \right)^{\gamma - \mu} \cdot \left[ \int_{E(c, i)} \mathbb{D}(c^*, i, \varepsilon) \frac{\partial A(c^*, i, \varepsilon)}{\partial c^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right) d\varepsilon, \right.
\]

\[
- \frac{1 + r^f}{s(1 - \gamma)(1 + \pi) g'(L)} \cdot \frac{\phi \left( \frac{\mathcal{E}(c^*, i) + \sigma^2/2}{\sigma} \right)}{1 - \Phi \left( \frac{\mathcal{E}(c^*, i) + \sigma^2/2}{\sigma} \right)} \cdot \mathbb{C}(c^*, i) \left. \right]
\]

(D.48)

or can we write

\[
0 = \sigma (1 - \beta) \left( c^* \right)^{-\mu} + \beta \mathbb{E} \left( \mathcal{V}(a', \eta'; \mathcal{G}') \right)^{\gamma - \mu} \cdot \left[ \int_{E(c, i)} \mathbb{D}(c^*, i, \varepsilon) \frac{\partial A(c^*, i, \varepsilon)}{\partial c^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right) d\varepsilon
\]

\[
- \frac{1 + r^f}{s(1 - \gamma)(1 + \pi) g'(L)} \cdot m \left( -\frac{\mathcal{E}(c^*, i) + \sigma^2/2}{\sigma} \right) \cdot \mathbb{C}(c^*, i) \right],
\]

(D.49)

where \( m(\bullet) = \phi(\bullet)/\Phi(\bullet) \) is the inverse Mills ratio. When \( c^* + i \leq a \), the first-order condition (D.48) and (D.49) becomes

\[
0 = \sigma (1 - \beta) \left( c^* \right)^{-\mu} + \beta \mathbb{E} \left( \mathcal{V}(a', \eta'; \mathcal{G}') \right)^{\gamma - \mu} \cdot \mathbb{D}(c^*, i, \varepsilon) \frac{\partial A(c^*, i, \varepsilon)}{\partial c^*} \phi \left( \frac{\varepsilon + \sigma^2/2}{\sigma} \right).
\]

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D.3.3 The envelope condition for the state

The equation (D.31) and (D.44) defines a nonlinear equation system for \( c^* \in C^+ \) and \( i^* \in I^+ \). In the numerical exercise, the non-linear equation system will be solved. To solve this, one needs to know \( \mathbb{V}(a', \eta; \mathcal{G}) \) as well as its derivative with respect to \( a' \). The value function \( \mathbb{V}(\bullet) \) can be updated through value function iterations. But we still need a way to update the \( \frac{\partial \mathbb{V}(\bullet)}{\partial a} \). In the next proposition, we use the envelope condition to formulate the \( \frac{\partial \mathbb{V}(\bullet)}{\partial a} \), which can be used to update the partial derivative of the value.

**Proposition D.11** Suppose that the value function \( \mathbb{V}(a, \eta; \mathcal{G}) \) is continuously differentiable in \( a \). Given \( a > 0 \) and \( \eta \), if \( c \in C^+ \) and \( i^* \in I^+ \) both exist,

\[
\frac{\partial \mathbb{V}(a, \eta; \mathcal{G})}{\partial a} = (1 - \beta)(c^*)^{1-\mu} \cdot \mathbb{V}(a, \eta; \mathcal{G})^{1-\mu}. \tag{D.50}
\]

**Proof** It is not hard to notice that

\[
\frac{\partial \mathbb{E}(c, i)}{\partial a} = -\frac{\partial \mathbb{E}(c, i)}{\partial c}. \tag{D.51}
\]

Moreover, we have \( \frac{\partial A(\bullet)}{\partial a} = -\frac{\partial A(\bullet)}{\partial c} \). Therefore, it is easy to see that

\[
\frac{\partial}{\partial a} \left( \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon \right) = -\frac{\partial}{\partial c} \left( \lambda_{\eta H} \int_{\varepsilon} \mathbb{V}(a', \eta; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon \right). \tag{D.52}
\]

In the optimum, the first-order derivative of the value with respect to the state is

\[
\frac{\partial \mathbb{V}(a, \eta; \mathcal{G})}{\partial a} = \frac{\beta}{1 - \gamma} \cdot \mathbb{V}(a, \eta; \mathcal{G})^{1-\mu} \cdot \mathbb{E} \left( \mathbb{V}(a', \eta'; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon + \lambda_{\eta L} \int_{\varepsilon} \mathbb{V}(a', 0; \mathcal{G}')^{1-\gamma} f(\varepsilon) \, d\varepsilon \right). \tag{D.52}
\]

By the condition (D.52), we can apply the equation (D.46) to write the equation above into the envelope condition (D.50). ■

D.4 Numerical Algorithm

On the individual-level, we define two evenly distributed grid vector of cash-on-hand \( A^H \in [w, a_{\text{sup}}] \) and \( A^L \in [w, a_{\text{sup}}] \) for the H-type and L-type, respectively. On each grid point, we define a search grid of consumption \( C \in [\bar{c}a, \overline{\chi}a] \) to perform the global search for the optimal level of consumption, where
\( \chi \) is close to 0 and \( \overline{\chi} \) is close to 1. Let \( G_0 \) be the initial guess of the vector of the aggregate state, and let \( V^{H}_0(a, G_0) \) and \( V^{L}_0(a, G_0) \) denote the initial guesses for the value functions of the H- and L-type, respectively. Moreover, we let \( C^{H}_0(a, G_0) \) and \( C^{L}_0(a, G_0) \) denote the initial guesses of consumption for the H- and L-type, respectively. On each grid point on \( A^H \) or \( A^L \) and on each search point on \( C \), the optimal investment (given consumption) can be obtained by solving the non-linear equation (D.31). The term of \( \frac{\partial V(\bullet)}{\partial a} \) in the non-linear equation (D.31) can be obtained through the envelop condition (D.50). On each grid point of the cash-on-hand, we find the consumption \( C^{H}_1(a, G_0) \) (\( C^{L}_1(a, G_0) \)), such that \( C^{H}_1(a, G_0) \) (\( C^{L}_1(a, G_0) \)) and the implies optimal investment correspond to the maximal level of the value \( V^{H}_1(a, G_0) \) (\( V^{L}_1(a, G_0) \)). We repeat this procedure until both values and consumptions converge. Values and consumptions are interpolated using Akima cubic-splines. Expected continuation values are computed using Gauss–Hermite quadrature points and weights.

After obtaining household’s optimal policies, we iterate over the distribution of cash-on-hand in the economy. For this purpose, we define a much larger grid \( A^H \in [w, a^{\sup}] \) and \( A^L \in [w, a^{\sup}] \) for the H- and L-type, respectively. As initial guesses, we assign equal probability for each grid point on \( A^H \) (\( A^L \)). Using household’s policy functions and the probability distribution, we calculate the probability that households end up on each grid point of \( A^H \) (\( A^L \)) in the next period. We iterate until the probability distribution converges. The aggregate state \( G_0 \) can thus be updated to \( G_1 \), based on the probability distribution and household’s policy functions. Repeat the steps above, until aggregate state \( G \) converges.
References


